Part IA — Differential Equations

Based on lectures by Prof. A. Challinor

Michaelmas 2021

Syllabus

Basic calculus
Informal treatment of differentiation as a limit, the chain rule, Leibnitz's rule, Taylor series, informal treatment of O and o notation and l'Hôpital's rule; integration as an area, fundamental theorem of calculus, integration by substitution and parts. [3]

Informal treatment of partial derivatives, geometrical interpretation, statement (only) of symmetry of mixed partial derivatives, chain rule, implicit differentiation. Informal treatment of differentials, including exact differentials. Differentiation of an integral with respect to a parameter.[2]

First-order linear differential equations
Equations with constant coefficients: exponential growth, comparison with discrete equations, series solution; modelling examples including radioactive decay.

Equations with non-constant coefficients: solution by integrating factor. [2]

Nonlinear first-order equations
Separable equations. Exact equations. Sketching solution trajectories. Equilibrium solutions, stability by perturbation; examples, including logistic equation and chemical kinetics. Discrete equations: equilibrium solutions, stability; examples including the logistic map. [4]

Higher-order linear differential equations
Complementary function and particular integral, linear independence, Wronskian (for second-order equations), Abel's theorem. Equations with constant coefficients and examples including radioactive sequences, comparison in simple cases with difference equations, reduction of order, resonance, transients, damping. Homogeneous equations. Response to step and impulse function inputs; introduction to the notions of the Heaviside step-function and the Dirac delta-function. Series solutions including statement only of the need for the logarithmic solution. [8]

Multivariate functions: applications
Directional derivatives and the gradient vector. Statement of Taylor series for functions on $\mathbb{R}^n$. Local extrema of real functions, classification using the Hessian matrix. Coupled first

order systems: equivalence to single higher order equations; solution by matrix methods. Non-degenerate phase portraits local to equilibrium points; stability.

Simple examples of first- and second-order partial differential equations, solution of the wave equation in the form $f(x + ct) + g(x - ct)$. [5]

Contents

§0 Introduction Video

24 lecture course. Full notes will be provided on moodle, small typefont indicates an aside, either because the material is non examinable or will be covered in greater detail in a later course.

Four example sheets.

§0.1 Schedule

1. Basic Calculus (5 lectures)
2. First-order linear differential equations (2)
3. Nonlinear first-order differential equations (4)
4. Higher-order linear differential equations (8)
5. Multivariate functions: applications (5)

§0.2 Introduction

They describe the rate of change of the dependent variable wrt the independent variable.

Example 0.1 (Newton's 2nd law)

$$m\frac{d^2x}{dt^2} = F$$

If F depends only on t, then we can simply integrate twice. However, if F is a function of x (such as a charged particle in a electric field which varied over space).

Applied course - emphasises methods and results rather than proof or existence.

§0.3 Limits

- Informally, if $\lim_{x\to x_0} f(x) = A$, then $f(x)$ can be made arbitrarily close to A by making x sufficiently close to x_0
  - Note, does not require $f(x_0)$ to equal A (or even to exist) – a limit is a statement about the behaviour of a function in the vicinity of x_0, but not at that point.
- More formally, for a function $f(x)$ defined on some open interval containing x_0 (but not necessarily at x_0), $\lim_{x\to x_0} f(x) = A$ means that
  - for any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x)-A| < \epsilon$ for all $0 < |x-x_0| < \delta$.

- Right hand limit, for example, defined similarly but with $0 < |x - x_0| < \delta$ replaced with $0 < x - x_0 < \delta$. A similar procedure can be done for left hand limits

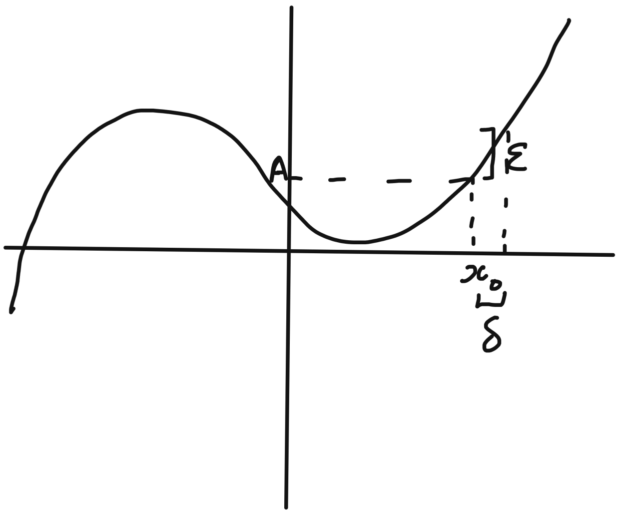


Figure 1: Right hand limit

- We can also define limits at infinity, e.g. $\lim_{x \to x_0} f(x) = A$ means that
  - for any $\epsilon > 0$, there exists $X > 0$ such that $|f(x) - A| < \epsilon$ for all $x > X$.

§0.3.1 Properties

- If $f(x)$ has a limit at a point, it is unique
- If $\lim_{x \to x_0} f(x) = A$ and $\lim_{x \to x_0} g(x) = B$, then:
  - $\lim_{x \to x_0}[f(x) + g(x)] = A + B$
  - $\lim_{x \to x_0}[f(x)g(x)] = AB$
  - $\lim_{x \to x_0}[f(x)/g(x)] = A/B$. If $B = 0$, the limit of the quotient does not exist if $A \neq 0$, but may exist in the indeterminate case $A = B = 0$

These properties will be proved carefully in the Analysis 1 course next term, but will be used as without proof in this course.

§0.3.2 Proof of uniqueness of limits

Suppose that $\lim_{x \to x_0} f(x) = A$ and $\lim_{x \to x_0} f(x) = B$. In terms of our epsilon-delta definition, this means that for any $\epsilon > 0$ there exists $\delta_A > 0$ and $\delta_B > 0$ such that

for $0 < |x - x_0| < \delta_A,\ |f(x) - A| < \epsilon/2$, where $\epsilon/2$ is an arbitrary positive quantity.
and for $0 < |x - x_0| < \delta_A,\ |f(x) - B| < \epsilon/2$

Now let $\delta = min(\delta_A, \delta_B)$ and consider $0 < |x - x_0| < \delta$ - follows that

$$\begin{aligned}|A - B| &= |[A - f(x)] - [B - f(x)]| \\ &\leq |A - f(x)| + |B - f(x)| \\ &\leq \epsilon\end{aligned}$$

Since this holds for all $\epsilon > 0$, we must have $A = B$.

I. BASIC CALCULUS

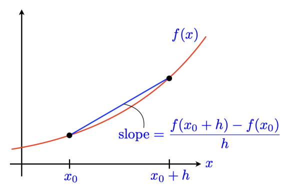
1 Differentiation

We begin with a recap of the main ideas of differentiation. Much of the material in this section will likely already be familiar to you.

Definition (Derivative of a function). We define the *derivative* of a function $f(x)$ with respect to its argument x as the *function* given by the *limit*

$$\frac{df}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \,. \tag{1}$$

As shown in the figure to the right, the value of df/dx at argument $x = x_0$ is the slope of the graph of $f(x)$ at the point x_0, and so determines the rate of change of $f(x)$ with respect to x there.

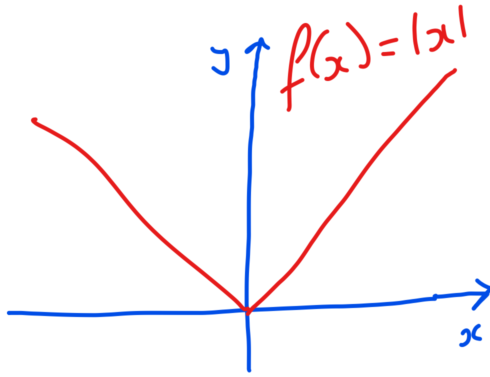


For the function $f(x)$ to be *differentiable* at the point x_0, and so for the function df/dx to be well-defined there, the left-hand limit (i.e., h is negative and approaches zero from below) and the right-hand limit (h is positive and approaches zero from above) must be defined and equal:

$$\lim_{h\to 0^-} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h\to 0^+} \frac{f(x_0+h) - f(x_0)}{h} \,.$$

This is actually quite a strong restriction on the "smoothness" of the function $f(x)$. As an example, $f(x) = |x|$ is not differentiable at $x = 0$ since

$$\lim_{h\to 0^-} \frac{|h|}{h} = -1 \quad \text{but} \quad \lim_{h\to 0^+} \frac{|h|}{h} = +1 \,.$$



Aside: limits

You will see much more about limits next term in *Analysis I*.

Informally, if $\lim_{x\to x_0} f(x) = A$, then $f(x)$ can be made arbitrarily close to A by making x sufficiently close to x_0. Note that we do *not* require $f(x_0)$ to equal A (or even to exist) – the limit is a statement about the behaviour of the function in the vicinity of x_0, but not actually at x_0.

A little more formally, for a function $f(x)$ defined on some open interval containing x_0 (but not necessarily at x_0), $\lim_{x\to x_0} f(x) = A$ means that

> for any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - A| < \epsilon$ for all $0 < |x - x_0| < \delta$.

The right-hand limit, for example, is defined similarly but with $0 < |x - x_0| < \delta$ replaced by $0 < x - x_0 < \delta$.

We can also define limits at infinity. For example, $\lim_{x\to\infty} f(x) = A$ means that

> for any $\epsilon > 0$, there exists $X > 0$ such that $|f(x) - A| < \epsilon$ for all $x > X$.

Various properties of limits will be proven in *Analysis I*. Here, we simply state some of the most important properties.

- If $f(x)$ has a limit at a point, it is unique.
- If $\lim_{x\to x_0} f(x) = A$ and $\lim_{x\to x_0} g(x) = B$, then:
  - $\lim_{x\to x_0}[f(x) + g(x)] = A + B$;
  - $\lim_{x\to x_0}[f(x)g(x)] = AB$;
  - $\lim_{x\to x_0}[f(x)/g(x)] = A/B$ for $B \neq 0$.

If $B = 0$, the limit of the quotient does not exist if $A \neq 0$, but *may* exist in the *indeterminate* case $A = B = 0$.

The notation df/dx for the derivative of a function is due to Leibniz. Notice how the denominator shows what the argument of the function is. Other widely used notations for the derivative include $f'(x)$, due to Lagrange, and $\dot{f}$, due to Newton and usually reserved for differentiation with respect to time.

For sufficiently smooth functions, we can define higher derivatives recursively. For example, for the second derivative

$$\frac{d}{dt}\left(\frac{df}{dt}\right) = \frac{d^2 f}{dt^2} = f''(t) = \ddot{f}(t)\,.$$

For the nth derivative, the notation

$$\frac{d^n f}{dx^n} = f^{(n)}(x)$$

is sometimes used.

1.1 Big-O and little-o notation

Two very useful concepts in applied mathematics are big O (pronounced "Oh") and little o, which is sometimes written $\underline{o}$ to distinguish clearly between the two symbols.

These two concepts, sometimes called *order parameters*, are used to give comparative scalings between functions sufficiently close to some limiting point x_0 (which may be ∞).

Definition (O and o notations).

1. $f(x)$ is $o[g(x)]$ as $x \to x_0$ if

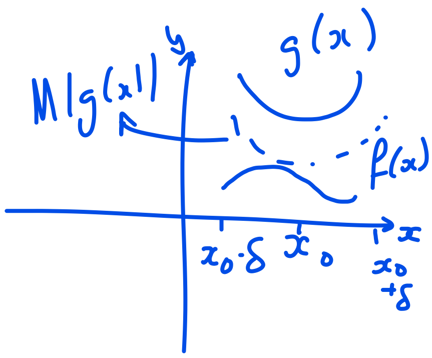
$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = 0\,,$$

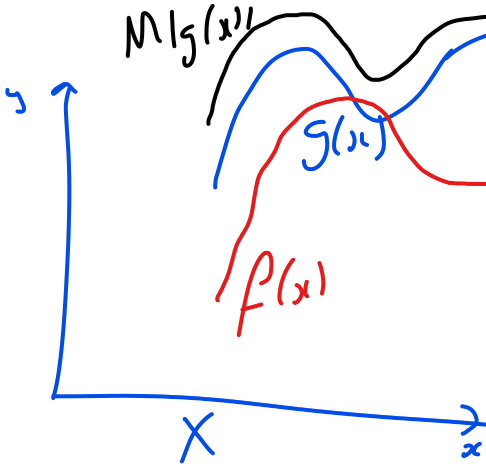
   commonly written as $f(x) = o[g(x)]$. Informally, this means that "$f(x)$ is much smaller than $g(x)$ as $x \to x_0$".

2. $f(x)$ is $O[g(x)]$ as $x \to x_0$ if $f(x)/g(x)$ is bounded as $x \to x_0$, i.e., there exists $\delta > 0$ and $M > 0$ such that for all x with $0 < |x - x_0| < \delta$,

$$|f(x)| \le M|g(x)|\,.$$

   This is commonly written as $f(x) = O[g(x)]$.





These ideas can be extended to the behaviour at infinity. For example, $f(x)$ is $O[g(x)]$ as $x \to \infty$ if there exists $M > 0$ and $X > 0$ such that for all $x > X$ we have $|f(x)| \leq M|g(x)|$.

Note that there is an abuse of notation here when writing, for example, $f(x) = O[g(x)]$, since $O[g(x)]$ is not a function. Rather, we mean that $f(x)$ and $g(x)$ are in a class of functions with the required property of varying in a particular way as x_0 is approached. Writing $f(x) \in O[g(x)]$ is more appropriate, but $f(x) = O[g(x)]$ is commonplace.

Note from the definitions, $f = o(g) \Rightarrow f = O(g)$ but not vice versa. For example, if $f(x) = 2x$, we have $f(x) = O(x)$ since $f(x)/x = 2$, but then $f(x) \neq o(x)$.

Example. The function x^2 is $o(x)$ as $x \to 0$ since

$$\lim_{x\to 0} \frac{x^2}{x} = \lim_{x \to 0} x = 0\,.$$

Example. The function $x^2 + x$ is $O(x^2)$ as $x \to \infty$. This follows since for $x > 1$ we have $x^2 > x$, so that

$$|x^2 + x| < 2|x^2| \quad \text{for } x > 1\,.$$

Generally, a polynomial with largest power x^n will be $O(x^n)$ (or any larger power of x) as $x \to \infty$.

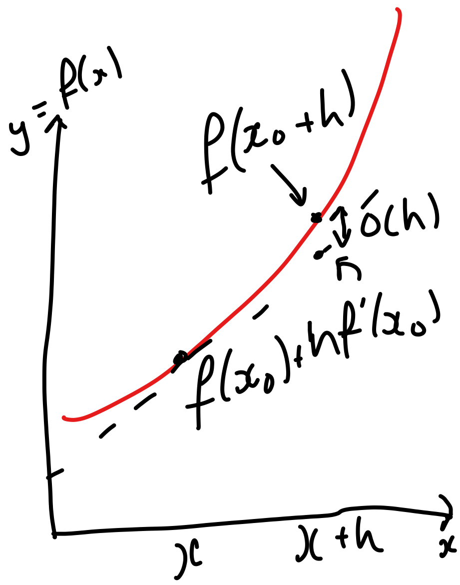
Some further examples:

- $x = o(\sqrt{x})$ as $x \to 0$;
- $\sin 2x = O(x)$ as $x \to 0$ as $\sin 2x \approx 2x$ for small x;
- $\sqrt{x} = o(x)$ as $x \to \infty$; and
- $\cos(x) = O(1)$ for all x as $|\cos x| \leq 1$.

Order parameters are frequently used in calculus to classify remainder terms before taking a limit. For example, we can write Eq. (1) as

$$\left.\frac{df}{dx}\right|_{x_0} = \frac{f(x_0 + h) - f(x_0)}{h} + \frac{o(h)}{h} \quad \text{as } h \to 0\,. \qquad (2)$$

We can establish the truth of this by taking the limit $h \to 0$ of both sides. The left-hand side does not depend on h, while the limit of the first term on the right is, by definition, df/dx at x_0. It follows that the limit of the (final) remainder term, $\lim h_{\to 0}\, o(h)/h = 0$, which is consistent with the little-o notation.



Multiplying Eq. (2) across by h, we obtain

$$f(x_0+h) = f(x_0) + h\,\frac{df}{dx}\bigg|_{x_0} + o(h) \quad \text{as } h \to 0\,. \qquad (3)$$

This identifies $f(x_0+h)$ with the value given by the tangent line at the point x_0 plus a remainder that is $o(h)$.

1.2 Rules of differentiation

Let us remind ourselves of the several useful rules of differentiation, and how they arise from the fundamental definitions presented above.

1.2.1 Chain rule

Consider the case where we want to differentiate a "function of a function" of the independent variable, i.e., $f(x) = F(g(x))$. For example, we might have $f(x) = \sin(x^2 - x + 2)$, where $F(X) = \sin(X)$ and $g(x) = x^2 - x + 2$.

Theorem (Chain rule). Given $f(x) = F(g(x))$, then

$$\frac{df}{dx} = F'(g(x))\,\frac{dg}{dx} = \frac{dF}{dg}\frac{dg}{dx}\,. \qquad (4)$$

The first term on the right is the derivatve of the function F with respect to its argument, evaluated at $g(x)$.

For our specific example,

$$\frac{d}{dx}\sin(x^2 - x + 2) = \left[\cos(x^2 - x + 2)\right](2x - 1).$$

Proof (Chain rule). We have

$$\begin{aligned}\frac{df}{dx} &= \lim_{h\to 0} \frac{F\left(g(x+h)\right) - F\left(g(x)\right)}{h} \\ &= \lim_{h\to 0} \frac{F\left(g(x) + hg'(x) + o(h)\right) - F\left(g(x)\right)}{h}\,,\end{aligned}$$

where we assume that g is differentiable. Now, if we write

$$X = g(x) \quad \text{and} \quad H = hg'(x) + o(h)\,,$$

then

$$\begin{aligned}F\big(g(x) + hg'(x) + o(h)\big) &= F(X+H) \\ &= F(X) + HF'(X) + o(H)\,,\end{aligned}$$

so that

$$\begin{aligned}\frac{df}{dx} &= \lim_{h\to 0} \frac{\left[hg'(x) + o(h)\right]F'(X) + o(H)}{h} \\ &= g'(x)F'(X) + \lim_{h\to 0} \frac{o(H)}{h}\,.\end{aligned}$$

The final term on the right is zero. To see this, consider the following two cases separately.

- $g'(x) = 0$. In this case, $H = o(h)$ as $h \to 0$ and so goes to zero "faster" than h as $h \to 0$. It follows that a term of $o(H)$ as $H \to 0$ goes to zero "faster still" as $h \to 0$ and so is certainly $o(h)$.
- $g'(x) \neq 0$. In this case, $H = O(h)$ but is certainly not $o(h)$ as $h \to 0$. This means that for small enough h, H is proportional to h and so a term of $o(H)$ goes to zero faster than linearly in h and so is $o(h)$ as $h \to 0$.

Finally, we recover the chain rule:

$$\frac{df}{dx} = g'(x)F'(g(x))\,.$$

1.2.2 Product rule

Consider the situation where $f(x) = u(x)v(x)$, i.e., f can be written as the product of two other functions u and v.

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Theorem (Product rule). Given $f(x) = u(x)v(x)$,

$$\frac{df}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}\,. \tag{5}$$

The proof of this is an exercise on *Examples Sheet 1.*

The *quotient rule* is a special case of the product rule, replacing $v \to 1/v$:

$$f = \frac{u}{v} \to f' = \frac{u'v - v'u}{v^2}\,. \tag{6}$$

1.2.3 Leibniz's rule

The product rule can be generalized to higher-order derivatives very straightforwardly by recursive application. Considering $f(x) = u(x)v(x)$, we have

$$\begin{aligned}
f' &= u'v + uv'\,,\\
f'' &= u''v + u'v' + u'v' + uv''\\
&= u''v + 2u'v' + uv''\,,\\
f''' &= u'''v + u''v' + 2u''v' + 2u'v'' + u'v'' + uv'''\\
&= u'''v + 3u''v' + 3u'v'' + uv'''\,.
\end{aligned}$$

This should be reminscent to you of Pascal's triangle and the binomial theorem.

Theorem (Leibniz's rule). Given $f(x) = u(x)v(x)$,

$$\begin{aligned}
f^{(n)}(x) &= \sum_{r=0}^{n} \binom{n}{r} u^{(n-r)}v^{(r)} \qquad (7)\\
&= u^{(n)}v + nu^{(n-1)}v' + \frac{n(n-1)}{2!}u^{(n-2)}v''\\
&+ \cdots + \frac{n!}{m!(n-m)!}u^{(n-m)}v^{(m)} + \cdots + uv^{(n)}\,.
\end{aligned}$$

Here, recall, a superscript (n) denotes the nth derivative (with, for example, $u^{(0)} = u$), and the binomial coefficient,

$$\binom{n}{r} \equiv \frac{n!}{r!(n-r)!}\,, \tag{8}$$

denotes the number of combinations of r elements that can be taken from n elements without replacement.

Of course, the Leibniz rule relies on the functions u and v being n-times differentiable.

Proof (Leibniz's rule). We prove this by induction. Equation (7) reduces to the product rule when $n = 1$, and so is true in this case. We now show that it is true for $n + 1$ if true for $n \geq 1$.

Differentiating the Leibniz rule for n with the product rule, we have

$$\begin{aligned} f^{(n+1)} &= \frac{d}{dx} \sum_{r=0}^{n} \binom{n}{r} u^{(n-r)} v^{(r)} \\ &= \sum_{r=0}^{n} \binom{n}{r} \left[u^{(n+1-r)} v^{(r)} + u^{(n-r)} v^{(r+1)} \right] . \end{aligned} \tag{9}$$

The last term on the right is

$$\begin{aligned} \sum_{r=0}^{n} \binom{n}{r} u^{(n-r)} v^{(r+1)} &= \sum_{r=0}^{n-1} \binom{n}{r} u^{(n-r)} v^{(r+1)} + u v^{(n+1)} \\ &= \sum_{r=1}^{n} \binom{n}{r-1} u^{(n+1-r)} v^{(r)} + u v^{(n+1)} , \end{aligned}$$

where we have relabelled $r \to r - 1$ in passing to the second line. Combining with the first term on the right of Eq. (9), we have

$$\begin{aligned} f^{(n+1)} &= u^{(n+1)} v + \sum_{r=1}^{n} \left[\binom{n}{r} + \binom{n}{r-1} \right] u^{(n+1-r)} v^{(r)} + u v^{(n+1)} \\ &= u^{(n+1)} v + \sum_{r=1}^{n} \binom{n+1}{r} u^{(n+1-r)} v^{(r)} + u v^{(n+1)} \\ &= \sum_{r=0}^{n+1} \binom{n+1}{r} u^{(n+1-r)} v^{(r)} . \end{aligned} \tag{10}$$

Here, we have used *Pascal's rule*,

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1} ,$$

which follows from Pascal's triangle (or directly from the expression for the binomial coefficients, Eq. 8).

Equation (10) is Leibniz's rule for $n + 1$, proving that if true for n, it is also true for $n + 1$.

1.3 Taylor series

1.3.1 Taylor's theorem

Recall Eq. (3), which can be rewritten as

$$f(x_0+h) = f(x_0) + h\,\frac{df}{dx}\bigg|_{x=x_0} + o(h) \quad \text{as } h \to 0\,.$$

Provided the first n derivatives of f exist, this can be generalised to *Taylor's theorem.*

Theorem (Taylor's theorem). For n-times differentiable $f(x)$, we have

$$f(x_0+h) = f(x_0) + h\,\frac{df}{dx}\bigg|_{x=x_0} + \frac{h^2}{2!}\,\frac{d^2 f}{dx^2}\bigg|_{x=x_0} + \cdots + \frac{h^n}{n!}\,\frac{d^n f}{dx^n}\bigg|_{x=x_0} + E_n\,, \quad (11)$$

where $E_n = o(h^n)$ as $h \to 0$.

In fact, if $f^{(n+1)}$ exists $\forall x \in (x_0, x_0+h)$ and $f^{(n)}$ is continuous in this range, it can be shown (see *Analysis I*) that $E_n = O(h^{n+1})$ as $h \to 0$. In particular,

$$E_n = \frac{f^{(n+1)}(x_n)}{(n+1)!}h^{n+1}\,, \quad (12)$$

for some x_n with $x_0 \le x_n \le x_0 + h$.

Note that $E_n = O(h^{n+1})$ is a stronger statement than $E_n = o(h^n)$. For example, for $0 < a < 1$,

$$h^{n+a} = o(h^n) \quad \text{as } h \to 0\,,$$

but

$$h^{n+a} \neq O(h^{n+1}) \quad \text{as } h \to 0\,.$$

Taylor's theorem is an exact statement that expresses the value of a function f at a point $x_0 + h$ in terms of the value of the function at x_0, its derivatives at x_0, and an error term E_n whose behaviour we know as h gets smaller.

1.3.2 Taylor Polynomials

If we write $x = x_0 + h$, then Eq. (11) can be rewritten as

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \cdots + \frac{(x - x_0)^n}{n!} f^{(n)}(x_0) + E_n \,. \quad (13)$$

The first $n+1$ terms on the right-hand side form the nth-order *Taylor polynomial* of $f(x)$ about the point $x = x_0$. Note that the coefficients of this polynomial are such that its first n derivatives match those of $f(x)$ at x_0.

The Taylor polynomial can be used to approximate functions in the vicinity of a point, with an error controlled by E_n. Note that this is a local approximation and does not necessarily tell us anything about the function far from the point (although it sometimes does).

Regarding the nth Taylor polynomial as a series, if the limit $n \to \infty$ exists (i.e., the series converges) we obtain the *Taylor series* of $f(x)$ about the point x_0.

1.4 L'Hôpital's rule

Taylor series representations are very useful to understand L'Hôpital's rule, which can be used to determine the value of limits of *indeterminate forms*.

Theorem (L'Hôpital's rule). Let $f(x)$ and $g(x)$ be differentiable at $x = x_0$, with continuous first derivatives there, and

$$\lim_{x \to x_0} f(x) = f(x_0) = 0 \quad \text{and} \quad \lim_{x \to x_0} g(x) = g(x_0) = 0 \,.$$

Then if $g'(x_0) \neq 0$,

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)} \,, \quad (14)$$

provided the limit on the right exists.

Proof. For this special case (the rule actually applies in much more general circumstances), we have

$$\begin{aligned} f(x) &= f(x_0) + (x - x_0)f'(x_0) + o(x - x_0) \\ &= 0 + (x - x_0)f'(x_0) + o(x - x_0)\,, \\ g(x) &= g(x_0) + (x - x_0)g'(x_0) + o(x - x_0) \\ &= 0 + (x - x_0)g'(x_0) + o(x - x_0)\,. \end{aligned}$$

lim f' + o(x - x0)/(x - x0) / lim g' + o(x - x0)/(x - x0)

It follows that

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f' + o(x - x_0)/(x - x_0)}{g' + o(x - x_0)/(x - x_0)} = \frac{f'(x_0)}{g'(x_0)}\,,$$

where in the last step we have used $g'(x_0) \neq 0$. Finally, since the first derivatives were assumed continuous at $x = x_0$, we have

$$\frac{f'(x_0)}{g'(x_0)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}\,.$$

The rule can be generalised to higher orders. For example, if $f(x_0) = f'(x_0) = 0$ and if $g(x_0) = g'(x_0) = 0$, then

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f''(x)}{g''(x)}\,,$$

by just applying l'hôpital again.

provided that the limit exists.

As a concrete example, consider

$$f(x) = 3\sin x - \sin 3x \quad \text{and} \quad g(x) = 2x - \sin 2x\,.$$

For these functions, $f(0) = g(0) = f'(0) = g'(0) = f''(0) = g''(0) = 0$. As an exercise, show that

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = 3 = \lim_{x\to 0} \frac{f'''(x)}{g'''(x)}\,.$$

2 Integration

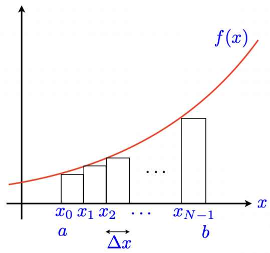
You will be familiar with integration as the "area under a curve" and also as the inverse of differentiation. We shall review both concepts in this section, as well as recapping some useful methods for integrating functions.

2.1 Integrals as Riemann sums

The "area under the curve" concept of an integral can be formalised as a *Riemann sum*. This is another topic that will be dealt with in detail in *Analysis I*, but let us briefly explore the idea here.

Definition (Integral). The *integral* of a (suitably well-defined) function $f(x)$ is the limit of a sum, e.g.,

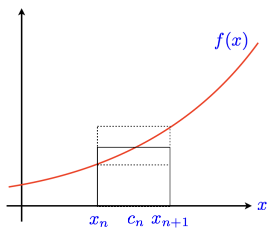
$$\int_a^b f(x)\,dx \equiv \lim_{\Delta x\to 0}\sum_{n=0}^{N-1} f(x_n)\Delta x = \lim_{N\to\infty}\sum_{n=0}^{N-1} f(x_n)\Delta x\,, \tag{15}$$



where $\Delta x = (b-a)/N$ and $x_n = a + n\Delta x$, as shown in the figure to the right.

The essence of the Riemann sum definition of the integral is that the limit should not depend exactly on how the rectangles are chosen (e.g., their widths do not have to be uniform, provided they all go to zero as $N \to 0$).

We now demonstrate that for sufficiently well-behaved functions, Eq. (15) coincides with the familar "area under the curve". We do this by first considering each rectangle in turn for finite N. Provided that $f(x)$ is continuous, the area under the curve between x_n and x_{n+1} is



$$A_n = (x_{n+1} - x_n) f(c_n)\,,$$

where $x_n \le c_n \le x_{n+1}$. This is an example of the *mean-value theorem*. We will not prove this here, but see the figure to the right for why this is plausible.

If $f(x)$ is differentiable, we have

$$\begin{aligned} f(c_n) &= f(x_n) + O(c_n - x_n) \quad \text{as } c_n - x_n \to 0 \\ &= f(x_n) + O(\Delta x) \quad (\text{since } c_n \le x_n + \Delta x)\,, \end{aligned}$$

from which it follows that

$$A_n = f(x_n)\Delta x + O(\Delta x^2) \quad \text{as } \Delta x \to 0\,.$$

Finally, the total area under the curve between $x = a$ and $x = b$ is

$$\begin{aligned}
A &= \lim_{N\to\infty} \sum_{n=0}^{N-1} A_n \\
&= \lim_{N\to\infty} \sum_{n=0}^{N-1} \left[f(x_n)\Delta x + O(\Delta x^2)\right] \\
&= \lim_{N\to\infty} \sum_{n=0}^{N-1} f(x_n)\Delta x + \lim_{N\to\infty} NO\left(\frac{(b-a)^2}{N^2}\right) \\
&= \lim_{N\to\infty} \sum_{n=0}^{N-1} f(x_n)\Delta x + \lim_{N\to\infty} O\left(\frac{(b-a)^2}{N}\right) \\
&= \lim_{N\to\infty} \sum_{n=0}^{N-1} f(x_n)\Delta x = \int_a^b f(x)\, dx\,.
\end{aligned}$$

2.2 Fundamental theorem of calculus

The concept of integration as the inverse of differentation is formulated in the *fundamental theorem of calculus.*

Theorem (Fundamental theorem of calculus). Let $F(x)$ be defined as

$$F(x) = \int_a^x f(t)\, dt\,.$$

Then

$$\frac{dF}{dx} = f(x)\,. \tag{16}$$

Proof. From the definition of the derivative, we have

$$\frac{dF}{dx} = \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)\, dt - \int_a^x f(t)\, dt\right].$$

The two overlapping parts cancel, and so

$$\begin{aligned}\frac{dF}{dx} &= \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt \\ &= \lim_{h\to 0}\frac{1}{h}\left[f(x)h + O(h^2)\right] \\ &= \lim_{h\to 0}\left[f(x)+O(h)\right] \\ &= f(x)\,,\end{aligned}$$

← Same argument as before. A = h f(cn) = . . .

where we used the mean-value theorem in passing to the second line.

Note that another way to interpret the Fundamental theorem of calculus is that the integral $F(x)$ is the solution of the differential equation

$$\frac{dF}{dx} = f(x) \quad \text{with } F(a) = 0\,. \tag{17}$$

As corollaries to the Fundamental theorem of calculus, we have

$$\frac{d}{dx}\int_x^b f(t)\,dt = -f(x)\,,$$

and, using the chain rule,

$$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(g(x))g'(x)\,.$$

Notation. We write *indefinite integrals* either as $\int f(x)\,dx$ or $\int^x f(t)\,dt$, where the unspecified lower limit gives rise to an integration constant.

2.3 Methods of integration

Integration is more difficult than differentiation. We cannot always evaluate integrals analytically in terms of simple (or not so simple!) functions. However, for those cases where we can, tricks such as integration by substitution or by parts are often helpful.

Identity	Term in integrand	Substitution
$\cos^2\theta + \sin^2\theta = 1$	$\sqrt{1-x^2}$	$x = \sin\theta$
$1 + \tan^2\theta = \sec^2\theta$	$1 + x^2$	$x = \tan\theta$ or sinh u
$\cosh^2 u - \sinh^2 u = 1$	$\sqrt{x^2-1}$	$x = \cosh u$
$\cosh^2 u - \sinh^2 u = 1$	$\sqrt{1+x^2}$	$x = \sinh u$
$1 - \tanh^2 u = \text{sech}^2 u$	$1 - x^2$	$x = \tanh u$ or sin θ
	$x^2 - 1$	$x = \sec\theta$ or cosh u

Table 1: Useful trigonometric or hyperblic substitutions.

2.3.1 Integration by substitution

If the integrand contains a function of a function, integration by substitution is often useful.

Example. Consider

$$I = \int \frac{1-2x}{\sqrt{x-x^2}}\,dx\,.$$

Let $u = x - x^2$ so that $du = (1-2x)dx$; then

$$I = \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + c = 2\sqrt{x-x^2} + c\,,$$

where c is a constant of integration.

Trigonometric (or hyperbolic) substitutions are often useful. If the function in the second column of Table 1 appears in the integrand, try proceeding by making the substitution in the third column and simplify using the idenity in the first column.

Example. Consider

$$I = \int \sqrt{2x - x^2}\,dx\,.$$

Since $2x - x^2 = 1 - (x-1)^2$, let us try $x - 1 = \sin\theta$ so that $dx = \cos\theta d\theta$. It follows that

$$\begin{aligned}
I &= \int \cos^2\theta\, d\theta \\
&= \frac{1}{2}\int (1+\cos 2\theta)\ d\theta \\
&= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + c \\
&= \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + c \\
&= \frac{1}{2}\arcsin(x-1) + \frac{1}{2}(x-1)\sqrt{1-(x-1)^2} + c\,,
\end{aligned}$$

where c is an integration constant and arcsin is the inverse sine function.

2.4 Integration by parts

"Integration by parts" exploits the product rule, which we write here in the form

$$uv' = (uv)' - u'v\,.$$

Integrating both sides gives rise to the following.

Theorem (Integration by parts). For functions u and v,

$$\int uv'\, dx = uv - \int u'v\, dx. \tag{18}$$

Example. Consider

$$I = \int_0^\infty xe^{-x}\, dx\,.$$

Let $u = x$ and $v' = e^{-x}$, so that $v = -e^{-x}$. We then have

$$\begin{aligned}
I &= \left[-xe^{-x}\right]_0^\infty + \int_0^\infty e^{-x}\, dx \\
&= 0 + \left[-e^{-x}\right]_0^\infty = 1\,.
\end{aligned}$$

Example. Consider

$$I = \int \ln x\, dx\,.$$

Let $u = \ln x$ and $v' = 1$, so that $v = x$. Integrating by parts gives

$$\begin{aligned} I &= x \ln x - \int x \left(\frac{1}{x}\right) dx \\ &= x \ln x - x + c\,, \end{aligned}$$

where c is an integration constant.

3 Partial differentiation

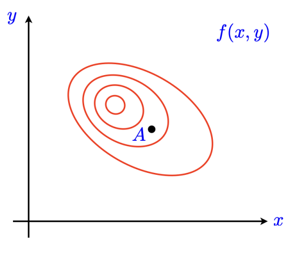
Here we generalize differentiation to functions of more than one variable.

3.1 Functions of several variables

Multivariate functions depend on more than one independent variable. Some physical examples include:

- the height of some terrain, which depends on both latitude and longitude;
- the density of air in this room, which depends on both position and time; and
- the energy of a thermodynamic system, which depends on its volume and temperature.

Considering a function of two variables, $f(x, y)$, we can represent it as a *contour plot* (see figure to the right) where f is constant on each contour line.



What is the slope of the function f at the point A? The answer obviously depends on the direction. The first thing to do is to figure out the slope along directions parallel to the coordinate axes, which leads us to the concept of *partial differentiation*.

3.2 Partial derivatives

Definition (Partial derivative). Given a function of several variables, for example, $f(x, y)$, the *partial derivative* of f with respect to x is the rate of change of f as x varies at fixed y. It is given by

$$\left.\frac{\partial f}{\partial x}\right|_y = \lim_{\delta x \to 0} \frac{f(x + \delta x, y) - f(x, y)}{\delta x} . \tag{19}$$

Note that the partial derivative of f with respect to x essentially corresponds to the slope of f experienced when moving purely left to right (in the positive x-direction).

Similarly, the partial derivative of $f(x, y)$ with respect to y is defined as the function

$$\left.\frac{\partial f}{\partial y}\right|_x = \lim_{\delta y \to 0} \frac{f(x, y + \delta y) - f(x, y)}{\delta y} , \tag{20}$$

i.e., the slope of f experienced when moving in the positive y-direction.

Note the "curly" partial derivative symbol, ∂, to distinguish from the ordinary derivative.

Example. Consider $f(x, y) = x^2 + y^3 + e^{xy^2}$. To compute the partial derivatives with respect to x, we simply hold y constant and differentiate regularly as if x were the only variable:

$$\left.\frac{\partial f}{\partial x}\right|_y = 2x + y^2 e^{xy^2} .$$

Similarly, for the partial derivative with respect to y,

$$\left.\frac{\partial f}{\partial y}\right|_x = 3y^2 + 2xye^{xy^2} .$$

We can also compute second derivatives

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_y = 2 + y^4 e^{xy^2} ,$$

$$\left.\frac{\partial^2 f}{\partial y^2}\right|_x = 6y + 2xe^{xy^2} + 4x^2 y^2 e^{xy^2} ,$$

as well as *mixed partial derivatives*

$$\frac{\partial}{\partial x}\left(\left.\frac{\partial f}{\partial y}\right|_x\right)\bigg|_y = 2ye^{xy^2} + 2xy^3 e^{xy^2} \,,$$
$$\frac{\partial}{\partial y}\left(\left.\frac{\partial f}{\partial x}\right|_y\right)\bigg|_x = 2ye^{xy^2} + 2xy^3 e^{xy^2} \,.$$

It is necessary to be careful to avoid ambiguity in which arguments specifically are being held fixed when the partial derivatives are being taken. However, it is often cumbersome to indicate this explicitly. In cases where *all* other variables are being held fixed, we shall omit the $|_y$, for example, and simply write $\partial f/\partial x$.

With this convention, for example,

$$\frac{\partial}{\partial x}\left(\left.\frac{\partial f}{\partial y}\right|_x\right)\bigg|_y = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} \,.$$

Notice that for the function in the example above,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} \,, \tag{21}$$

and so it does not matter in which order the partial derivatives are taken. This can be shown to hold true generally (*Schwarz's theorem* or *Clairaut's theorem*), provided that the function has continuous (mixed) second derivatives at the point of interest.

Finally, we note that an alternative subscript notation is sometimes used for partial derivatives. For example,

$$f_x \equiv \frac{\partial f}{\partial x} \,; \quad f_{xy} \equiv \frac{\partial^2 f}{\partial y \partial x} .$$

Note the ordering in the second case. The left-hand side should be interpreted as $(f_x)_y$, so that the partial derivative is first taken with respect to x before the resulting function is differentiated with respect to y. Of course, in most cases this is not important since $f_{xy} = f_{yx}$.

3.3 Multivariate chain rule

The chain rule (Eq. 4) tells us how to differentiate a "function of a function". How does this extend to multivariate functions? For example, for a given path $x(t)$ and $y(t)$, where t is a parameter along the path, the function $f(x, y)$ can be considered a function of t, i.e., $f((x(t), y(t))$. How do we calculate df/dt?

Consider the change in $f(x, y)$ under an arbitrary small displacement in any direction,

$$(x, y) \to (x + \delta x, y + \delta y)\,.$$

We have

$$\begin{aligned}\delta f &= f(x + \delta x, y + \delta y) - f(x, y)\\ &= [f(x + \delta x, y + \delta y) - f(x + \delta x, y)]\\ &\quad + [f(x + \delta x, y) - f(x, y)]\;.\end{aligned}$$

Taylor expanding the second term in square brackets, we have

$$f(x + \delta x, y) - f(x, y) = f_x(x, y)\delta x + o(\delta x)\,.$$

Similarly,

$$f(x+\delta x, y+\delta y) - f(x+\delta x, y) = f_y(x+\delta x, y)\delta y + o(\delta y)\,.$$

This involves the partial derivative f_y evaluated at $(x + \delta x, y)$. We can expand this about the point (x, y) using

$$f_y(x + \delta x, y) = f_y(x, y) + f_{yx}(x, y)\delta x + o(\delta x)\,.$$

Putting this together, we find

$$\begin{aligned}\delta f = {}& [f_y(x, y) + f_{yx}(x, y)\delta x + o(\delta x)]\,\delta y + o(\delta y)\\ & + f_x(x, y)\delta x + o(\delta x)\,. \qquad (22)\end{aligned}$$

Taking the limit as $\delta x, \delta y \to 0$, and defining the *differential* of f as

$$df = \lim_{\delta x, \delta y \to 0} \delta f\,,$$

we obtain the multivariate chain rule in differential form.

Theorem (Chain rule for partial derivatives). The differential df of the function f is related to the differentials of its arguments dx and dy as

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy\,. \tag{23}$$

Note that the remaining terms in Eq. (22) all go to zero faster than δx or δy in the limit $\delta x, \delta y \to 0$.

We can now use the multivariate chain rule to calculate df/dt along the path $(x(t), y(t))$. Dividing by δt in Eq. (22) and then taking the limit, we have

$$\frac{d}{dt} f(x(t), y(t)) = \lim_{\delta t \to 0} \frac{\delta f}{\delta t} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}\,. \tag{24}$$

Note that, since $x(t)$ and $y(t)$ are functions of t alone, their *ordinary* derivatives appear in this expression.

Another commonly occuring case is where the path is parameterised by one of the coordinates. For example, y may be specified as $y(x)$ so that along the path we have $f(x, y(x))$, which is a function of x. The rate of change of this function with respect to x is

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}\,. \tag{25}$$

Note that f changes both because the first argument of the function (x) is changing *and* because the second argument (y) changes as x changes.

3.3.1 Integral form of the chain rule

The chain rule, Eq. (23), can be integrated along a path to get the change in the function f between the start- and end-points:

$$\Delta f = \int df = \int \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \right)\,.$$

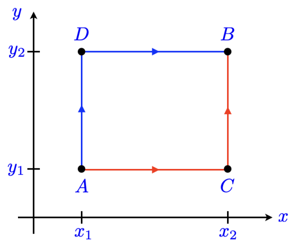
$\Delta f = f(x(t), y(t)) - f(x, y)$.

If the path is parameterised as $(x(t), y(t))$, we have

$$\Delta f = \int \frac{df}{dt}\, dt = \int \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \right) dt\,.$$

For given start- and end-points, the result will not depend on the particular path that is chosen.

As an example, consider integrating from point $A = (x_1, y_1)$ to $B = (x_2, y_2)$. If we consider the two paths shown in the figure to the right, for that going via $C = (x_2, y_1)$, we have



$$\begin{aligned} \Delta f &= \int_{x_1}^{x_2} \frac{\partial f}{\partial x}(x, y_1)\, dx + \int_{y_1}^{y_2} \frac{\partial f}{\partial y}(x_2, y)\, dy \\ &= [f(x_2, y_1) - f(x_1, y_1)] + [f(x_2, y_2) - f(x_2, y_1)] \\ &= f(x_2, y_2) - f(x_1, y_1)\,. \end{aligned}$$

For the path going via $D = (x_1, y_2)$, we have

$$\begin{aligned} \Delta f &= \int_{y_1}^{y_2} \frac{\partial f}{\partial y}(x_1, y)\, dy + \int_{x_1}^{x_2} \frac{\partial f}{\partial x}(x, y_2)\, dx \\ &= [f(x_1, y_2) - f(x_1, y_1)] + [f(x_2, y_2) - f(x_1, y_2)] \\ &= f(x_2, y_2) - f(x_1, y_1)\,, \end{aligned}$$

which is the same as going via C.

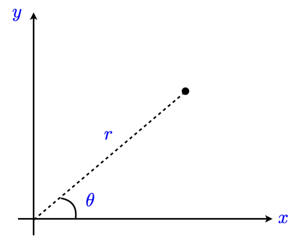
3.4 Applications of the multivariate chain rule

3.4.1 Change of variables

The chain rule naturally plays a central role when we change the independent variables, for example through a coordinate transformation.

Example. Consider transforming from Cartesian coordinates (x, y) to plane-polar coordinates (r, θ), with

$$x = r\cos\theta\,, \quad y = r\sin\theta\,. \tag{26}$$



The original function $f(x, y)$ can be thought of as a function of r and θ, i.e., $f(x(r, \theta), y(r, \theta))$. To compute the partial derivatives with respect to r and θ, we use the chain rule. For r, we have

$$\left.\frac{\partial f}{\partial r}\right|_\theta = \left.\frac{\partial f}{\partial x}\right|_y \left.\frac{\partial x}{\partial r}\right|_\theta + \left.\frac{\partial f}{\partial y}\right|_x \left.\frac{\partial y}{\partial r}\right|_\theta$$
$$= \left.\frac{\partial f}{\partial x}\right|_y \cos\theta + \left.\frac{\partial f}{\partial y}\right|_x \sin\theta\,,$$

and for θ,

$$\left.\frac{\partial f}{\partial \theta}\right|_r = \left.\frac{\partial f}{\partial x}\right|_y \left.\frac{\partial x}{\partial \theta}\right|_r + \left.\frac{\partial f}{\partial y}\right|_x \left.\frac{\partial y}{\partial \theta}\right|_r$$
$$= -\left.\frac{\partial f}{\partial x}\right|_y r\sin\theta + \left.\frac{\partial f}{\partial y}\right|_x r\cos\theta\,.$$

3.4.2 Implicit differentiation

Consider the expression $f(x, y, z) = c$, for some constant c. This defines a surface in 3D space, and so it *implicitly* defines a functional relationship between one of the coordinates x, y and z and the other two, i.e.,

$$z = z(x, y) \quad \text{or} \quad x = x(y, z) \quad \text{or} \quad y = y(x, z)\,.$$

Depending on the function $f(x, y, z)$, we may not be able to express these functional relationships in closed form. However, we can still evaluate their partial derivatives using implicit differentiation.

Example. Consider

$$xy + y^2 z + z^5 = 1\,. \tag{27}$$

Finding $x(y, z)$ is straightforward since x only appears linearly. To determine $y(x, z)$ we have to solve a quadratic equation. However, we cannot find $z(x, y)$ explicitly since this would require solving a quintic equation.

Despite this, we can still determine $\partial z/\partial x|_y$, for example, by taking the derivative of Eq. (27) with respect to x, holding y constant, using implicit differentiation:

$$y + y^2 \left.\frac{\partial z}{\partial x}\right|_y + 5z^4 \left.\frac{\partial z}{\partial x}\right|_y = 0\,,$$

so that

$$\left.\frac{\partial z}{\partial x}\right|_y = -\frac{y}{y^2 + 5z^4}\,.$$

Generally, given $f(x, y, z) = c$, the chain rule (extended to three variables) gives

$$0 = df = \left.\frac{\partial f}{\partial x}\right|_{y,z} dx + \left.\frac{\partial f}{\partial y}\right|_{x,z} dy + \left.\frac{\partial f}{\partial z}\right|_{x,y} dz\,.$$

Note that we cannot vary x, y and z independently as we must stay in the surface. We can find the rate of change of z with x at fixed y from

$$0 = \left.\frac{\partial f}{\partial x}\right|_{y,z} \underbrace{\left.\frac{\partial x}{\partial x}\right|_y}_{=1} + \left.\frac{\partial f}{\partial y}\right|_{x,z} \underbrace{\left.\frac{\partial y}{\partial x}\right|_y}_{=0} + \left.\frac{\partial f}{\partial z}\right|_{x,y} \left.\frac{\partial z}{\partial x}\right|_y\,.$$

Therefore

$$\left.\frac{\partial z}{\partial x}\right|_y = -\frac{\partial f/\partial x|_{y,z}}{\partial f/\partial z|_{x,y}}\,. \tag{28}$$

We can similarly find that

$$\left.\frac{\partial x}{\partial y}\right|_z = -\frac{\partial f/\partial y|_{x,z}}{\partial f/\partial x|_{y,z}}\,, \qquad \left.\frac{\partial y}{\partial z}\right|_x = -\frac{\partial f/\partial z|_{x,y}}{\partial f/\partial y|_{x,z}}\,,$$

and so the relation

$$\left.\frac{\partial x}{\partial y}\right|_z \left.\frac{\partial y}{\partial z}\right|_x \left.\frac{\partial z}{\partial x}\right|_y = -1\,.$$

Note that normal *reciprocal rules* apply for partial derivatives, *provided* the same variables are being held constant. For example, for $f(x, y, z) = c$, similarly to Eq. (28) we have

$$\left.\frac{\partial x}{\partial z}\right|_y = -\frac{\partial f/\partial z|_{x,y}}{\partial f/\partial x|_{y,z}}\,,$$

so that

$$\left.\frac{\partial z}{\partial x}\right|_y = \frac{1}{\partial x/\partial z|_y}\,.$$

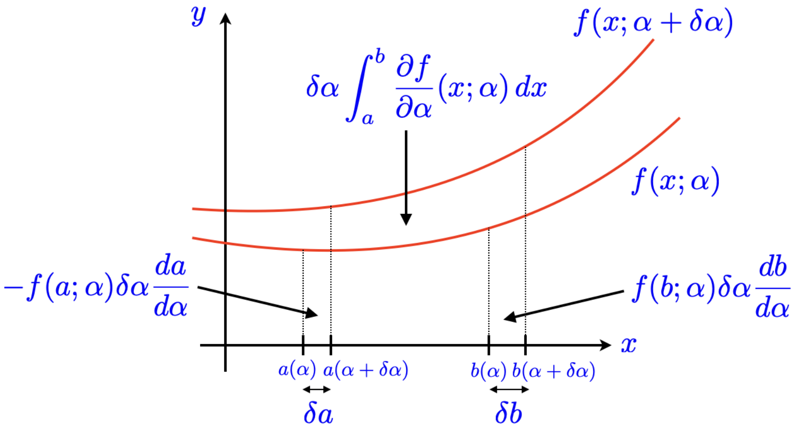


Figure 1: Contributions to the change in an integral when a parameter α appearing in the integrand, $f(x;\alpha)$, and the limits, $x = a(\alpha)$ and $x = b(\alpha)$, is varied.

3.4.3 Differentiation of an integral with respect to its parameters

Consider a family of functions $f(x;\alpha)$ where the parameter α labels the different members of the family. Define the integral

$$I(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x;\alpha)\,dx\,,$$

where we have allowed the limits of the integral to depend on the parameter also. What is $dI/d\alpha$?

Theorem (Differentiation of an integral w.r.t. a parameter). The derivative of $I(\alpha)$ is

$$\frac{d}{d\alpha}\int_{a(\alpha)}^{b(\alpha)} f(x;\alpha)\,dx = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial f}{\partial \alpha}(x;\alpha)\,dx + f(b;\alpha)\frac{db}{d\alpha} - f(a;\alpha)\frac{da}{d\alpha}\,. \quad (29)$$

Proof. We have

$$\frac{dI}{d\alpha} = \lim_{\delta\alpha\to 0}\frac{1}{\delta\alpha}\left[\int_{a(\alpha+\delta\alpha)}^{b(\alpha+\delta\alpha)} f(x;\alpha+\delta\alpha)\,dx - \int_{a(\alpha)}^{b(\alpha)} f(x;\alpha)\,dx\right]$$

$$= \lim_{\delta\alpha\to 0}\frac{1}{\delta\alpha}\left[\int_{a(\alpha)}^{b(\alpha)} f(x;\alpha+\delta\alpha) - f(x;\alpha)\,dx\right] + \lim_{\delta\alpha\to 0}\frac{1}{\delta\alpha}\int_{b(\alpha)}^{b(\alpha+\delta\alpha)} f(x;\alpha+\delta\alpha)\,dx - \lim_{\delta\alpha\to 0}\frac{1}{\delta\alpha}\int_{a(\alpha)}^{a(\alpha+\delta\alpha)} f(x;\alpha+\delta\alpha)\,dx\,. \qquad (30)$$

The first term on the right reduces to the integral of $\partial f/\partial\alpha$ between a and b. For the second term, we can use the mean-value theorem to write

$$\lim_{\delta\alpha\to 0}\frac{1}{\delta\alpha}\int_{b(\alpha)}^{b(\alpha+\delta\alpha)} f(x;\alpha+\delta\alpha)\,dx = \lim_{\delta\alpha\to 0}\left[f(\bar{x};\alpha+\delta\alpha) \times \left(\frac{b(\alpha+\delta\alpha)-b(\alpha)}{\delta\alpha}\right)\right],$$

where, inside the limit, $b(\alpha) \le \bar{x} \le b(\alpha+\delta\alpha)$. Taking the limit gives

$$\lim_{\delta\alpha\to 0}\frac{1}{\delta\alpha}\int_{b(\alpha)}^{b(\alpha+\delta\alpha)} f(x;\alpha+\delta\alpha)\,dx = f(b;\alpha)\frac{db}{d\alpha}\,.$$

The third term in Eq. (30) is handled similarly, and putting the three terms together establishes Eq. (29). The origin of these three terms is illustrated in Fig. 1.

Example. Consider

$$I(\lambda) = \int_0^{\lambda} e^{-\lambda x^2}\,dx\,.$$

Then

$$\frac{dI}{d\lambda} = e^{-\lambda^3} - \int_0^{\lambda} x^2 e^{-\lambda x^2}\,dx\,.$$

§1.1 Extra

If $f(x) = o(g(x))$ as $x \to x_0$, then $af(x) = o(g(x))$ as $x \to x_0$ for finite a.

Aside: Non examinable

Sketch proof of Taylor's Theorem. Start from FTC

$$\int_0^x f'(t)\,dt = f(x) - f(0)$$

$$\begin{aligned}
\implies f(x) &= f(0) + \int_0^x f'(t)\,dt \\
&= f(0) + \int_0^x \frac{d(t-x)}{dt} * f'(t)\,dt \\
&= f(0) + [(t-x)f'(t)]_{t=0}^{t=x} - \int_0^t (t-x)f''(t)\,dt \\
&= f(0) + xf'(0) - \frac{1}{2}\int_0^t \frac{d(t-x)^2}{dt} f''(t)\,dt \\
&= f(0) + xf'(0) - \frac{1}{2}[(t-x)^2 f''(t)]_{t=0}^{t=x} - \int_0^t + \frac{1}{2}\int_0^t (t-x)^2 f'''(t)\,dt \\
&= f(0) + xf'(0) + \frac{1}{2}x^2 f''(0) + \frac{1}{2}\int_0^t (t-x)^2 f'''(t)\,dt \\
&\vdots \\
&= f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0) + \frac{1}{n!}\int_0^x (t-x)^n f^{(n+1)}(t)\,dt
\end{aligned}$$

By MVT the remainder = somehow

$$= \frac{x^{(n+1)}}{(n+1)!} f^{(n+1)}(x_n) \text{ where } x_n \in [0, x]$$

□

(q12, sheet 1)

i.

$$p(V, S):$$

$$dp = \left.\frac{\partial p}{\partial V}\right|_S dV + \left.\frac{\partial p}{\partial S}\right|_V dS$$

Actually $S = S(V, T)$

$$\text{so, } dS = \left.\frac{\partial S}{\partial V}\right|_T dV + \left.\frac{\partial S}{\partial T}\right|_V dT$$

$$\implies dp = \left(\frac{\partial p}{\partial V}\bigg|_S + \frac{\partial p}{\partial S}\bigg|_V \frac{\partial S}{\partial V}\bigg|_T \right) dV + \frac{\partial p}{\partial S}\bigg|_V \frac{\partial S}{\partial T}\bigg|_V dT$$

ii.

$$\text{We want } \frac{\partial U}{\partial V}\bigg|_T \text{ and } \frac{\partial U}{\partial T}\bigg|_V : U(V,T)$$

$$\begin{aligned} \text{We are given } dU &= TdS - pdV \\ &= \underbrace{T \frac{\partial S}{\partial T}\bigg|_V}_{\equiv \frac{\partial U}{\partial T}|_V} dT + \underbrace{\left(T \frac{\partial S}{\partial V}\bigg|_T - p \right)}_{\equiv \frac{\partial U}{\partial V}|_T} dV \end{aligned}$$

3.4.3 Differentiation of an integral with respect to its parameters

Example 1.1

Suppose we want to evaluate

$$\begin{aligned} \int_0^\infty x^n e^{-x}\, dx \text{ where n is an integer} \\ \text{Let } I(\lambda) = \int_0^\infty e^{-\lambda x}\, dx = \frac{1}{\lambda} \\ \frac{d^n}{d\lambda^n} I = \int_0^\infty (-x)^n e^{-\lambda x}\, dx = (-1)^n \frac{n!}{\lambda^{n+1}} \\ \text{set } \lambda = 1 \text{ and we get } \int_0^\infty x^n e^{-x}\, dx = n! \end{aligned}$$

II. FIRST-ORDER LINEAR DIFFERENTIAL EQUATIONS

We now start to investigate differential equations proper. As noted in the Introduction, a differential equation is an equation involving derivatives of the dependent variable with respect to the independent variable(s).

Unlike regular, algebraic equations, the solution of a differential equation is a *function* that satisfies the equation. To obtain a unique solution requires specifying further suitable *boundary conditions*.

In this part of the course we shall consider *first-order differential equations*, where the highest derivative that appears is, for example, dy/dx.

1 Exponential function

As we shall see, the exponential function plays a key role in the solution of linear, first-order equations (we shall define linearity shortly). We therefore begin with a brief recap of the properties of the exponential function.

Definition (Exponential function). The exponential function is defined by the infinite series

$$\begin{aligned} \exp(x) &\equiv 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \,. \end{aligned} \tag{1}$$

Using the binomial theorem (see Examples Sheet 1 for an alternative approach) this can also be written as

$$\begin{aligned}\exp(x) &= \lim_{k\to\infty}\left(1+\frac{x}{k}\right)^k \\ &= \lim_{k\to\infty}\left[1+k\left(\frac{x}{k}\right)+\frac{k(k-1)}{2!}\left(\frac{x}{k}\right)^2+\cdots\right] \\ &= 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots .\end{aligned} \tag{2}$$

Differentiating the series definition (1) term by term, we see that

$$\begin{aligned}\frac{d\exp(x)}{dx} &= 1+2\times\frac{x}{2!}+3\times\frac{x^2}{3!}+\cdots \\ &= \exp(x)\,.\end{aligned} \tag{3}$$

Since $\exp(0) = 1$, we can alternatively *define* the exponential function as the unique solution of the differential equation

$$\frac{df}{dx} = f(x) \quad \text{with } f(0) = 1.$$

It follows that

$$\int_1^{\exp(x)} \frac{dy}{y} = x\,. \tag{4}$$

From this follows one of the main properties of exponentials:

$$\exp(x_1 + x_2) = \exp(x_1)\exp(x_2)\,. \tag{5}$$

To see this, note that, from Eq. (4),

$$x_1 + x_2 = \int_1^{\exp(x_1)} \frac{dy}{y} + \int_1^{\exp(x_2)} \frac{dy}{y}\,.$$

If we now make the variable substitution $u = \exp(x_1)y$ in the second term on the right, we have

$$\begin{aligned}x_1 + x_2 &= \int_1^{\exp(x_1)} \frac{dy}{y} + \int_{\exp(x_1)}^{\exp(x_1)\exp(x_2)} \frac{du}{u} \\ &= \int_1^{\exp(x_1)\exp(x_2)} \frac{dy}{y}\,.\end{aligned} \tag{6}$$

However, from Eq. (4) we also have

$$x_1 + x_2 = \int_1^{\exp(x_1+x_2)} \frac{dy}{y} \, .$$

Since the right-hand side of this expression must equal the right-hand side of Eq. (6) for all x_1 and x_2, we establish the property (5). (It may also be shown rather more directly from the limit definition in Eq. 2.)

The property in Eq. (5) is reminiscent of powers. Combining with $\exp(0) = 1$, we can write

$$\exp(x) = e^x \, , \tag{7}$$

where the value of e is

$$e = \exp(1) = \lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^k = 2.718\ldots \, .$$

The inverse function of $\exp(x)$ is denoted $\ln(x)$, so that

$$\exp(\ln x) = e^{\ln x} = x \, .$$

This is sometime written as the logarithm to the base-e, i.e., $\ln x = \log_e x$ and referred to as the *natural logarithm* of x.

The natural logarithm allows us to write[1] (for real $a > 0$)

$$a^x = \left(e^{\ln a}\right)^x = e^{x \ln a} \, ,$$

from which it follows that

$$\frac{da^x}{dx} = (\ln a) e^{x \ln a} = (\ln a) a^x \, .$$

The exponential function plays a prominent role in the analysis of differential equations since it is an *eigenfunction* of the derivative operator.

[1] While rational powers of a are defined directly in terms of repeated multiplication and roots, e.g., $a^{2/3}$ is the (positive) cube root of $a \times a$, this equation essentially defines what it means to raise a number to an irrational power. An alternative approach is to define a^x in terms of the limit of a sequence of terms a^{x_n}, where the x_n are rational but have x as their limit.

Definition (Eigenfunction). An *eigenfunction* of the derivative operator is a function that is unchanged, up to a multiplicative scaling by the *eigenvalue*, under the action of the operator. That is,

$$\frac{df}{dx} = \lambda f(x)\,,$$

where $f(x)$ is the eigenfunction and λ is the eigenvalue[2].

The eigenfunctions of d/dx are the functions $e^{\lambda x}$ since

$$\frac{d}{dx}e^{\lambda x} = \lambda e^{\lambda x}\,.$$

2 First-order linear differential equations

Differential equations of this form have the following properties.

- **Linear** – a differential equation is linear if the dependent variable, y say, and its derivatives only appear linearly.
- **First order** – a differential equation is first order if the higest derivative that appears is first order, i.e., dy/dx.

2.1 Homogeneous, first-order linear differential equations

We shall initially specialise further to consider *homogeneous* equations with *constant coefficients.*

- **Homogeneous** – a differential equation in which all terms involve the dependent variable (e.g., y) or its derivatives, so that $y = 0$ is a solution.
- **Constant coefficients** – a differential equation has constant coefficients if the independent variable (e.g., x) does not appear explicitly.

[2] The terminology "eigen" is from the German for "own".

Example. Consider the first-order, linear, homogeneous differential equation

$$5\frac{dy}{dx} - 3y = 0\,. \tag{8}$$

Let us try a solution $y = Ae^{\lambda x}$; then

$$\frac{dy}{dx} = A\lambda e^{\lambda x} = \lambda y\,,$$

so to be a solution we require $5\lambda - 3 = 0$. This is an example of a *characteristic equation* and the solution is $\lambda = 3/5$. Since this is a linear, homogeneous equation, the solution $y = Ae^{3x/5}$ holds for any A.

Generally, for any linear, homogeneous differential equation (so not necessarily first order), any constant multiple of a solution is also a solution.

Moreover, it can be shown that an nth-order linear differential equation has precisely n independent solutions. Specialising to the case of the first-order linear equation (8), we see that $y = Ae^{3x/5}$ is the *general solution.* A specific unique solution is obtained by specifying a suitable boundary condition for the dependent variable. For example, the value of y at $x = 0$ determines the constant A.

2.1.1 Discrete equations

It is interesting to compare the solution of Eq. (8) with that of a related *discrete equation.* A discrete equation involves a function evaluated at a discrete set of points.

Suppose we have $5dy/dx - 3y = 0$ and the boundary condition $y(0) = y_0$. We know that the solution is

$$y(x) = y_0 e^{3x/5}\,.$$

Consider now an approximate solution to this differential equation, whereby we approximate the derivative by a finite difference. In particular, consider discretising the equation at points $\{x_n\}$, spaced by h. The values of y at these points are $\{y_n\}$ and we approximate the derivative at x_n by

$$\left.\frac{dy}{dx}\right|_{x_n} \approx \frac{y_{n+1} - y_n}{h} \, .$$

(This is called the *forward Euler scheme* – it is not particularly good for numerical analysis and better schemes do exist, but it is fine to illustrate the key idea here.) The original equation (8) in discrete form becomes

$$5\left(\frac{y_{n+1} - y_n}{h}\right) - 3y_n \approx 0 \quad \Rightarrow \quad y_{n+1} \approx \left(1 + \frac{3h}{5}\right) y_n \, . \tag{9}$$

The final relation in Eq. (9) is an example of a *recurrence relation*. If we apply this repeatedly, we find

$$y_n = \left(1 + \frac{3h}{5}\right) y_{n-1} = \left(1 + \frac{3h}{5}\right)^2 y_{n-2} = \left(1 + \frac{3h}{5}\right)^n y_0 \, .$$

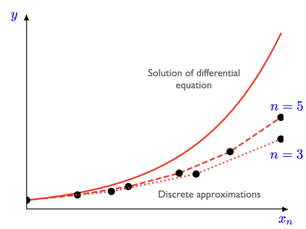
If we now suppose that $x_0 = 0$ and $x_n = nh = x$, i.e., we take n steps to go from $x = 0$ to the point of interest, x, we can write

$$y_n = y_0 \left(1 + \frac{3}{5}\frac{x}{n}\right)^n \, .$$

In the limit as $n \to \infty$, we expect this to agree with the exact solution $y(x) = y_0 e^{3x/5}$. This is indeed the case since, recalling Eq. (2), we have

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} y_0 \left(1 + \frac{3x/5}{n}\right)^n = y_0 \exp(3x/5) \, .$$

As shown in the figure to the right, the larger n is, the larger the value of y_n at the given point x.



2.1.2 Series solution

At this point, it is useful to introduce and illustrate a powerful technique for solving differential equations that we shall have much more to say about later in the course. The idea is to look for solutions in the form of an infinite power series,[3] i.e.,

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \,. \tag{10}$$

Substituting this expansion into the differential equation determines the coefficients a_n.

Example. Consider again

$$5\frac{dy}{dx} - 3y = 0\,. \tag{11}$$

Differentiating Eq. (10) gives

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1}\,.$$

Multiplying Eq. (11) by x (for convenience), we have terms involving

$$xy' = \sum_{n=1}^{\infty} n a_n x^n \,,$$
$$xy = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{m=1}^{\infty} a_{m-1} x^m \,,$$

where in the second line we have let $m = n + 1$. Relabelling $m \to n$ in this summation, and substituting into Eq. (11) (after multiplying through by x), we find

$$5\sum_{n=1}^{\infty} n a_n x^n - 3\sum_{n=1}^{\infty} a_{n-1} x^n = 0$$
$$\Rightarrow \quad \sum_{n=1}^{\infty} x^n \left(5 n a_n - 3 a_{n-1}\right) = 0\,.$$

[3] We shall see later that not all differential equations admit solutions of this form, but significant classes of equations do (or at least a close generalisation).

Since this must hold for all x, we must have

$$5na_n - 3a_{n-1} = 0 \quad \Rightarrow \quad a_n = \frac{3}{5n}a_{n-1} \quad (n \geq 1)\,.$$

Iterating this recursion relation, we have

$$a_n = \frac{3}{5n}a_{n-1} = \left(\frac{3}{5}\right)^2 \frac{1}{n(n-1)}a_{n-2} = \cdots = \left(\frac{3}{5}\right)^n \frac{1}{n!}a_0\,,$$

so that

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{3x}{5}\right)^n = a_0 e^{3x/5}\,.$$

In the final term we have identified the infinite series as the power series expansion of $\exp(3x/5)$; see Eq. (1).

2.2 Forced (inhomogeneous) equations

So far we have considered homogeneous differential equations. However, differential equations can also involve terms that are explicit functions of the independent variable and do not include the dependent variable nor its derivatives. Such equations are called *inhomogeneous* or *forced* equations and $y = 0$ is no longer a (trivial) solution.

We shall consider two simple, but important, types of forcing terms that break homogeneity.

2.2.1 Constant forcing

Constant forcing involves introducing a constant term in a differential equation.

Example. Consider

$$5\frac{dy}{dx} - 3y = 10\,,$$

where now we have added the constant term on the right-hand side. The general method for solving such forced, linear equations is as follows.

1. Find any solution of the forced equation. This is called the *particular integral* and we shall denote it by $y_p(x)$. This may require some guesswork. For our example, we might spot that there is a solution with $y =$ const., in which case we would have

$$y_p(x) = -10/3\,.$$

2. Now write the general solution in the form

$$y(x) = y_p(x) + y_c(x)\,,$$

involving the particular integral and a *complementary function* $y_c(x)$. Since the differential equation is linear in y and its derivatives, the complementary function must satisfy the homogeneous equation:

$$5\frac{dy_c}{dx} - 3y_c = 0 \quad \Rightarrow \quad y_c(x) = Ae^{3x/5}\,.$$

3. Combining, we have the full general solution

$$y(x) = -\frac{10}{3} + Ae^{3x/5}\,.$$

Any boundary conditions may now be applied (to the full solution y) to determine the constant A.

This method of solving linear, forced equations is general and is not restricted to first-order equations (with constant coefficients).

2.2.2 Eigenfunction forcing

A second particularly simple form of forcing is when the forcing is an eigenfunction of the underlying differential operator.

Example. Consider a radioactive material, in which isotope A decays into isotope B at a rate proportional to the number $a(t)$ of remaining nuclei of A, and B decays into C at a rate proportional to the number $b(t)$ of remaining nuclei of B. Determine $b(t)$.

We have

$$\frac{da}{dt} = -k_a a \,,$$
$$\frac{db}{dt} = k_a a - k_b b \,,$$

where k_a and k_b are the appropriate rate constants. We can solve the first equation for $a(t)$ directly to obtain

$$a(t) = a_0 e^{-k_a t} \,,$$

where a_0 is the number of A nuclei at $t = 0$. Substituting into the rate equation for $b(t)$ gives

$$\frac{db}{dt} + k_b b = k_a a_0 e^{-k_a t} \,. \qquad (12)$$

The forcing term in Eq. (12), being an exponential function, is an eigenfunction of the differential operator on the left-hand side. This suggests we try a particular integral

$$b_p = C e^{-k_a t},$$

for some suitable choice of the constant C. Substituting b_p into Eq. (12) we obtain

$$-k_a C + k_b C = k_a a_0 \quad \Rightarrow \quad C = \frac{k_a}{k_b - k_a} a_0 \,,$$

provided $k_a \neq k_b$.

As before, we then consider the general solution $b = b_c + b_p$, where b_c is the solution of the homogeneous equation:

$$\frac{db_c}{dt} + k_b b_c = 0 \quad \Rightarrow \quad b_c = D e^{-k_b t} \,,$$

for some constant D, and so

$$b(t) = \frac{k_a}{k_b - k_a} a_0 e^{-k_a t} + D e^{-k_b t} \,.$$

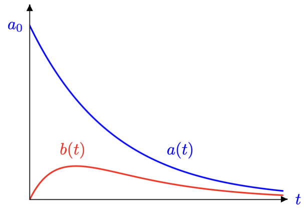
A particular situation of interest is when $b = 0$ at $t = 0$, i.e., the isotope B only appears due to decay of isotope

A. In this case,

$$b(t) = \frac{k_a}{k_b - k_a} a_0 \left(e^{-k_a t} - e^{-k_b t}\right) ,$$

$$\Rightarrow \quad \frac{b(t)}{a(t)} = \frac{k_a}{k_b - k_a} \left(1 - e^{(k_a - k_b)t}\right) .$$

Typical time variations of $a(t)$ and $b(t)$ are shown in the figure to the right. Analyses of this type allow rocks and other materials to be dated by measuring the ratio of isotopes (e.g., carbon dating).



The solution we obtained in this example is not valid for $k_a = k_b$. For this case, we could proceed by guessing a suitable alternative particular integral $b_p(t)$. However, we shall see below an alternative approach that will also inform the choice of a suitable particular integral.

2.3 Non-constant coefficients

So far we have considered linear, first-order differential equations with constant coefficients. We now drop the last restriction so that the coefficients of y and dy/dx are allowed to be functions of x.

Consider the general form of a first-order linear differential equation:

$$a(x)\frac{dy}{dx} + b(x)y = c(x) .$$

Dividing through by $a(x)$, we obtain the *standard form*:

$$\frac{dy}{dx} + p(x)y = f(x) . \tag{13}$$

We can always solve equations of this form by multiplying through by an *integrating factor* $\mu(x)$:

$$\mu\frac{dy}{dx} + (\mu p)y = \mu f . \tag{14}$$

The idea is to choose the integrating factor so that the left-hand side is the derivative $d(\mu y)/dx$ and the equation can be integrated directly. From the product rule, we require

$$\frac{d\mu}{dx} = \mu p \quad \Rightarrow \quad \frac{1}{\mu}\frac{d\mu}{dx} = p\,.$$

Integrating with respect to x we have

$$\int p\,dx = \int \frac{1}{\mu}\frac{d\mu}{dx}\,dx = \ln\mu\,.$$

Therefore, the integrating factor is

$$\mu(x) = \exp\left[\int^x p(u)du\right]\,, \qquad (15)$$

which is unique up to an irrelevant constant factor.

Since, by construction, Eq. (14) is equivalent to

$$\frac{d}{dx}\,(\mu y) = \mu f\,,$$

we have

$$\mu(x)y(x) = \int^x \mu(u)f(u)du\,,$$

from which $y(x)$ can be determined straightforwardly.

Example. Consider

$$x\frac{dy}{dx} + (1-x)y = 1\,,$$

or, in standard form,

$$\frac{dy}{dx} + \left(\frac{1}{x} - 1\right)y = \frac{1}{x}\,.$$

It follows that $p(x) = 1/x - 1$ and so, from Eq. (15),

$$\begin{aligned}\mu(x) &= \exp\left[\int^x p(u)\,du\right] \\ &= \exp\left[\int^x \left(\frac{1}{u} - 1\right)\,du\right] \\ &= \exp\left(\ln x - x\right) \\ &= xe^{-x}\,.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{d}{dx}\left(xe^{-x}y\right) &= e^{-x}\\ \Rightarrow \quad xe^{-x}y &= -e^{-x}+C\\ \Rightarrow \quad y &= -\frac{1}{x}+\frac{C}{x}e^{x}\,,\end{aligned}$$

where C is a constant to be determined from initial or boundary conditions.

In particular, if we require $y(x)$ to be finite as $x \to 0$ we must have $C = 1$, and so

$$y = \frac{e^x - 1}{x}\,.$$

2.3.1 Radioactive decay example revisited

It is instructive to reconsider the example of radioactive decay discussed above, now using the method of an integrating factor. In particular, this will allow us to handle easily the case $k_a = k_b$.

Recall Eq. (12), which we repeat here for convenience:

$$\frac{db}{dt} + k_b b = k_a a_0 e^{-k_a t}\,.$$

This is already in standard form, with $p(t) = k_b$. It follows that the integrating factor

$$\mu(t) = \exp\left[\int^t k_b du\right] = e^{k_b t}\,,$$

and so multiplying through in Eq. (12), we have

$$\frac{d}{dt}\left(e^{k_b t}b\right) = k_a a_0 e^{(k_b - k_a)t}\,. \tag{16}$$

We now consider the two cases, $k_a \neq k_b$ and $k_a = k_b$, separately.

1. If $k_a \neq k_b$, the right-hand side of Eq. (16) still varies with t. Integrating, we find

$$
\begin{aligned}
e^{k_b t} b &= \frac{k_a}{k_b - k_a} a_0 e^{(k_b - k_a)t} + D\,, \\
\Rightarrow \quad b(t) &= \frac{k_a}{k_b - k_a} a_0 e^{-k_a t} + D e^{-k_b t}\,,
\end{aligned}
$$

   exactly as before.

2. If $k_a = k_b = k$, the right-hand side of Eq. (16) is independent of t. Integrating in this case, we have

$$
\begin{aligned}
e^{kt} b &= k a_0 t + D\,, \\
\Rightarrow \quad b(t) &= k a_0 t e^{-kt} + D e^{-kt}\,.
\end{aligned}
$$

Note that in the case $k_a = k_b = k$, an appropriate particular integral of Eq. (12) is

$$
b_p(t) = k a_0 t e^{-kt}\,,
$$

rather than being simply $b_p(t) \propto e^{-kt}$.

§2.1 Extra

$$\begin{aligned}
\exp(x_1)\exp(x_2) &= \lim_{k\to\infty}\left(1+\frac{x_1}{k}\right)^k\left(1+\frac{x_2}{k}\right)^k \\
&= \lim_{k\to\infty}\left[\left(1+\frac{x_1}{k}\right)\left(1+\frac{x_2}{k}\right)\right]^k \\
&= \lim_{k\to\infty}\left[1+\frac{x_1+x_2}{k}+\frac{x_1x_2}{k^2}\right]^k \\
&= \lim_{k\to\infty}\left[1+\binom{k}{1}\frac{1}{k}\left(x_1+x_2+\frac{x_1x_2}{k}\right)+\dots\right]^k \\
&= \lim_{k\to\infty}\left[1+(x_1+x_2)+\frac{(x_1+x_2)^2}{2!}+\dots\right] \\
&= \exp(x_1+x_2)
\end{aligned}$$

III. NONLINEAR, FIRST-ORDER DIFFERENTIAL EQUATIONS

Having studied linear, first-order differential equations, we now consider *nonlinear* first-order equations. In this case, the dependent variable (e.g., $y(x)$) appears nonlinearly. Such equations have a rich phenomenology.

In general, a first-order differential equation takes the form

$$Q(x,y)\frac{dy}{dx} + P(x,y) = 0\,, \qquad (1)$$

for general, non-trivial functions P and Q. (This is not the most general form, since dy/dx could also appear nonlinearly, but we will not consider such cases here.)

1 Separable equations

Definition (Separable equation). A first-order differential equation is *separable* if it can be written in the form

$$q(y)dy = p(x)dx\,,$$

and so all the terms involving y explicitly can be collected to one side of the equation, and all the terms involving x explicitly can be collected to the other.

Separable equations can be solved directly by integration:

$$\int q(y)\,dy = \int p(x)\,dx\,.$$

Example. Consider

$$\left(x^2y - 3y\right)\frac{dy}{dx} - 2xy^2 = 4x\,.$$

Rearranging,

$$\frac{dy}{dx} = \frac{4x + 2xy^2}{x^2y - 3y} = \left(\frac{2x}{x^2-3}\right)\left(\frac{2+y^2}{y}\right).$$

Therefore

$$\begin{aligned} \frac{y}{2+y^2}dy &= \frac{2x}{x^2-3}dx\,, \\ \Rightarrow \quad \frac{1}{2}\ln\left(y^2+2\right) &= \ln\left(x^2-3\right)+C\,, \\ \Rightarrow \quad \left(y^2+2\right)^{1/2} &= A\left(x^2-3\right)\,, \end{aligned}$$

where C is an arbitrary integration constant and $A = e^C$.

2 Exact equations

Definition (Exact equation). Equation (1) is an *exact equation* if and only if the differential $P(x,y)dx + Q(x,y)dy$ is *exact*, i.e., there exists a function $f(x,y)$ such that

$$df = P(x,y)dx + Q(x,y)dy\,.$$

It follows that if Eq. (1) is exact, $df = 0$ and so $f(x,y) =$ const. is the solution. This is generally an implicit relation between x and y, which satisfies the differential equation.

If $P(x,y)dx + Q(x,y)dy$ is an exact differential of f, then $df = P(x,y)dx + Q(x,y)dy$. However, from the chain rule in differential form,

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\,,$$

so we must have

$$\frac{\partial f}{\partial x} = P \quad \text{and} \quad \frac{\partial f}{\partial y} = Q\,. \tag{2}$$

Solving these equations determines the function $f(x,y)$ (see the example below).

It follows from Eq. (2) that

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial P}{\partial y} \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}\,.$$

Since mixed second partial derivatives commute, we must have

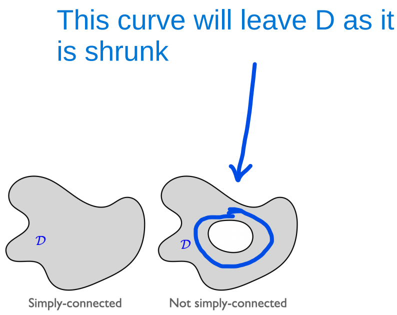
$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad (3)$$

if $P(x,y)dx + Q(x,y)dy$ is exact.

The converse is not necessarily true: it is possible for Eq. (3) to hold but $Pdx + Qdy$ not to be exact. Equation (3) is therefore a *necessary but not sufficient condition* for the differential to be exact. However, if it holds throughout some *simply-connected* domain, then it can be shown that the differential is exact in that domain.

Definition (Simply-connected domain). A domain $\mathcal{D}$ is simply-connected if it is path-connected (i.e., every pair of points can be connected by a path in $\mathcal{D}$) and any closed curve can be continuously shrunk to a point in $\mathcal{D}$ without leaving $\mathcal{D}$.

Examples. In 2D, a disk is simply-connected, but a disk with a hole in the middle is not (see the figure to the right for more general examples). The 2D surface of a sphere in 3D is simply-connected, but that of a torus (e.g., a ring doughnut) is not.



Theorem. If

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

throughout a simply connected domain $\mathcal{D}$, then $Pdx + Qdy$ is an exact differential of a single-valued function $f(x,y)$ in $\mathcal{D}$, i.e., there exists a single-valued function $f(x,y)$ in $\mathcal{D}$ such that $df = Pdx + Qdy$.

Aside: an inexact differential on a non-simply-connected domain

Consider the differential $Pdx + Qdy$ with

$$P = -\frac{y}{x^2+y^2} \quad \text{and} \quad Q = \frac{x}{x^2+y^2}\,.$$

We have

$$\frac{\partial P}{\partial y} = \frac{y^2 - x^2}{(x^2+y^2)^2} = \frac{\partial Q}{\partial x}\,,$$

so that Eq. (3) is satisifed away from the origin, $(x, y) = (0, 0)$. It follows that the differential is exact in a simply-connected region excluding the origin. Indeed, a suitable potential is $\theta(x, y)$, where θ is the polar angle of plane-polar coordinates, with

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta\,.$$

This follows since $\tan\theta = y/x$ gives, for example,

$$\frac{1}{\cos^2\theta}\frac{\partial\theta}{\partial x} = -\frac{y}{x^2} \quad \Rightarrow \quad \frac{\partial\theta}{\partial x} = -\frac{y}{x^2+y^2} = P.$$

However, the differential is *not* exact in the non-simply-connected region $0 < x^2 + y^2 \leq 1$ (which excludes the origin) since it cannot be written as the differential of a *single-valued* function throughout this domain. Rather, the potential θ changes by 2π in traversing any closed path that encircles the origin once.

Example. Consider

$$6y(y-x)\frac{dy}{dx} + 2x - 3y^2 = 0 \qquad (4)$$
$$\Rightarrow \quad \left(2x - 3y^2\right)dx + 6y(y-x)dy = 0\,,$$

so that

$$P(x, y) = 2x - 3y^2 \quad \text{and} \quad Q(x, y) = 6y^2 - 6xy\,.$$

It follows that

$$\frac{\partial P}{\partial y} = -6y = \frac{\partial Q}{\partial x}\,,$$

and so the differential $Pdx+Qdy$ is exact in any simply-connected domain.

Furthermore, the solution $f(x, y) =$ const. must satisfy the two equations

$$\frac{\partial f}{\partial x} = 2x - 3y^2 = P\,; \quad \frac{\partial f}{\partial y} = 6y^2 - 6xy = Q\,. \qquad (5)$$

If we integrate the first equation with respect to x, remembering that y is being held constant in the partial derivative $\partial f/\partial x$, we have

$$f(x, y) = x^2 - 3xy^2 + h(y)\,,$$

for some *function* $h(y)$. In general, this term is a function of y. If we take a partial derivative with respect to x keeping y constant, $h(y)$ will make no contribution.

Taking the partial derivative of this $f(x, y)$ with respect to y and comparing to the second equation in (5), we have

$$\begin{aligned} -6xy + \frac{dh}{dy} &= 6y^2 - 6xy \\ \Rightarrow \quad \frac{dh}{dy} &= 6y^2 \\ \Rightarrow \quad h(y) &= 2y^3 + C\,, \end{aligned}$$

for some constant C.

Therefore the solution to Eq. (4) is

$$f(x, y) = x^2 - 3xy^2 + 2y^3 = \text{const.}$$

This can, of course, be verified by direct substitution.

3 Solution curves and isoclines

It is not always possible to solve nonlinear equations explicitly, but we can gain insight into the "flow" of solutions using various graphical methods. We shall introduce some of these methods in this section.

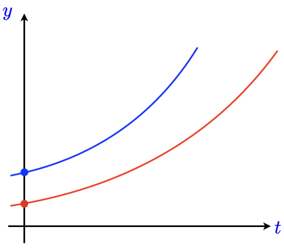
3.1 Solution curves

Consider a first-order differential equation of the form

$$\frac{dy}{dt} = f(t, y)\,.$$

Not necessarily t = 0

Each initial condition, e.g., specifying $y(0) = y_0$ at $t = 0$, will generate a distinct *solution curve* (or trajectory); see the figure to the right.



Example. Consider the nonlinear equation

$$\frac{dy}{dt} = t(1 - y^2)\,. \tag{6}$$

This equation is separable:

$$\frac{dy}{1-y^2} = t dt \,,$$

and can be integrated (using partial fractions is helpful) to obtain

$$\frac{1}{2}\ln\left|\frac{1+y}{1-y}\right| = \frac{1}{2}t^2 + C \,,$$

where C is a constant. Therefore

$$y = \frac{A - e^{-t^2}}{A + e^{-t^2}} \,, \tag{7}$$

for some further constant A (with $A = e^{2C}$ for $|y| < 1$, and $A = -e^{2C}$ for $|y| > 1$). This *general solution* of the differential equation produces a family of solution curves, parameterised by A.

We can express the parameter A in terms of, say, the value $y(0) = y_0$ using

$$y(0) = \frac{A-1}{A+1} \quad \Rightarrow \quad A = \frac{1+y_0}{1-y_0} \,.$$

The solution curves given by Eq. (7) are plotted in Fig. 1.

In this example, we could solve the differential equation exactly. However, let us now consider whether we can understand the key properties of the family of solution curves *without* solving the equation explicitly. This is important since we may not be able to solve a given (nonlinear) differential equation in closed form.

We first note, from Eq. (6), that $\dot{y} = 0$ for all t if $y = \pm 1$. There are therefore two constant solutions, $y = \pm 1$.

To proceed further, it is helpful to consider the *slope field* of the differential equation.

3.2 Slope field and isoclines

In the differential equation

$$\frac{dy}{dt} = f(t, y) \,,$$

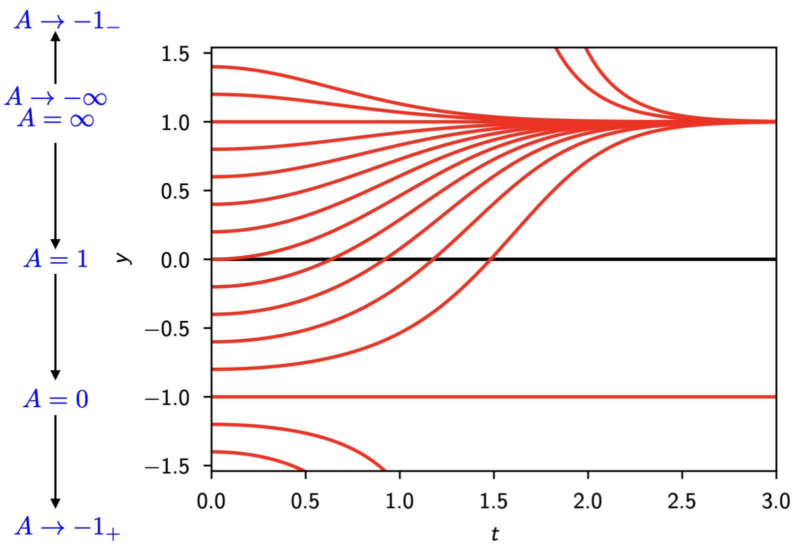


Figure 1: Solution curves $y(t)$ of the differential equation (6), as given by Eq. (7). The variation of the initial value y_0 with the parameter A is also indicated. Note that A changes discontinuously through $y_0 = 1$. Also, the solution curves are discontinuous where $e^{-t^2} = -A$, for A in the range $-1 < A < 0$.

the function $f(t, y)$ on the right-hand side determines the gradient (or slope) of the solution curve through the point (t, y). The *slope field* represents these gradients by short straight-line segments, one centred at each point (a regular grid in the t–y plane is often used), with gradient $f(t, y)$.

By construction, the slope field at a given point is tangent to the solution curve through that point. It therefore tells us the direction in which the solution curve flows.

It is often helpful to supplement the slope field with *isoclines*, which are curves along which $f(t, y)$ is constant.

For the example above, Eq. (6), we have

$$f(t, y) = t\left(1 - y^2\right)$$

so that, for $t > 0$, $\dot{y} < 0$ for $|y| > 1$ and $\dot{y} > 0$ for $|y| < 1$. The isoclines have

$$t\left(1 - y^2\right) = D \quad \Rightarrow \quad y^2 = 1 - D/t\,,$$

where D is a constant that parameterises the isoclines. Along each isocline, the slope field is constant; see Fig. 2.

By drawing curves through the slope field, we can construct approximations to the solution curves even if we cannot determine their functional form exactly.

Finally, note that if $f(t, y)$ is a single-valued function, the solution curves cannot cross in the t–y plane.

4 Fixed (equilibrium) points and stability

In this section, we consider the properties of *fixed points* or *equilibrium points* of differential equations. The analysis of fixed points typically reveals many important properties of the solution of the differential equation.

Definition (Fixed/equilibrium point). A *fixed point* or *equilibrium point* of a differential equation $dy/dt = f(t, y)$ is a constant solution, $y = c$. This corresponds to $dy/dt = 0$ for all t.

In the specific example above, Eq. (6), we have $f(t, y) = t(1 - y^2)$ and so there are fixed points at $y = \pm 1$. From consideration of the solution curves shown in Fig. 1, it is clear that these two fixed points have qualitatively different character.

Specifically, the solution curves converge towards $y = 1$ as t increases, while they diverge from $y = -1$. For these reasons, the fixed point $y = 1$ is said to be a *stable fixed point* while $y = -1$ is an *unstable fixed point.*

Definition (Stability of fixed points). A fixed point $y = c$ is *stable* if whenever y is deviated slightly from c, $y \to c$ as $t \to \infty$. A fixed point is *unstable* if the devitation grows in magnitude as $t \to \infty$.

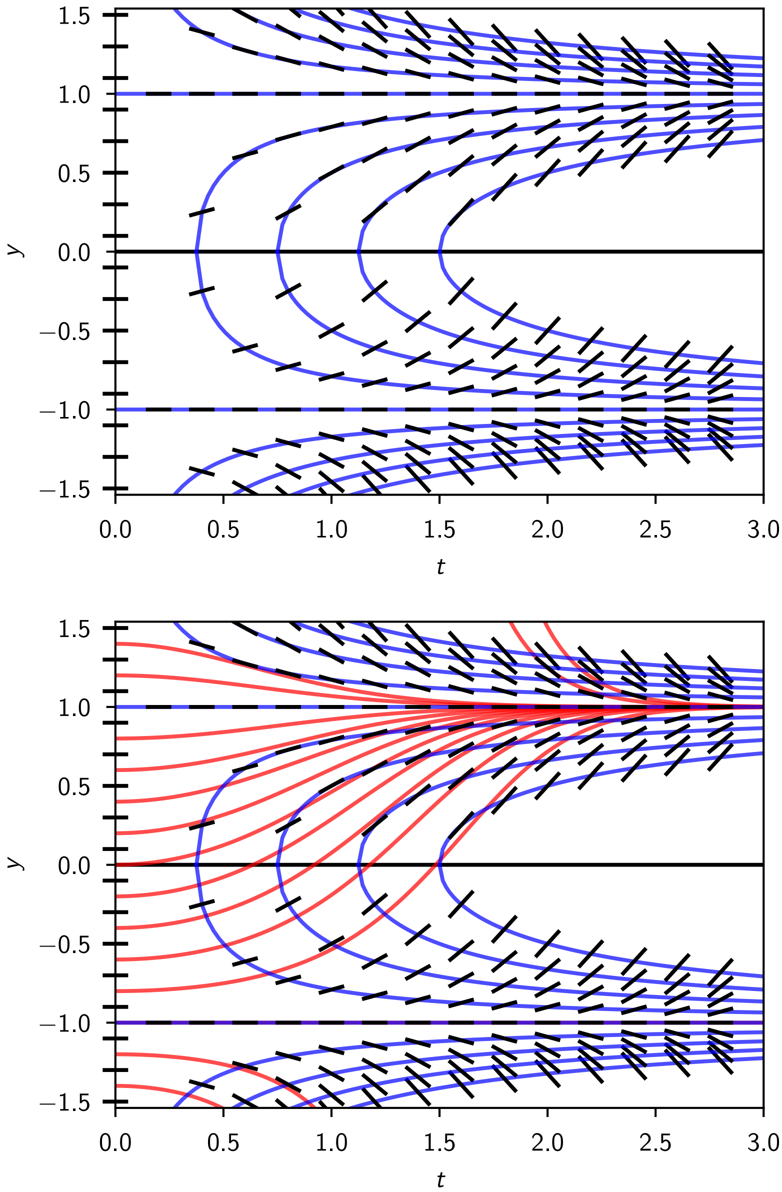


Figure 2: *Top*: isoclines (blue) and the slope field (black sticks) along these for the differential equation (6). *Bottom*: the solution curves (red) are tangent to the slope field everywhere.

4.1 Perturbation analysis and stability

To determine the stability of a fixed point we can use "perturbation analysis". This involves considering the form of the differential equation in the vicinity of the fixed point $y = c$.

Suppose that $y = c$ is a fixed point of the first-order differential equation $dy/dt = f(t, y)$, so that $f(t, c) = 0$ for all t. Consider a small perturbation from the fixed point, which we can write as

$$y(t) = c + \epsilon(t)\,,$$

where $\epsilon(t)$ is a small perturbation. Substituting into the differential equation, we have

$$\begin{aligned}\frac{d\epsilon}{dt} &= f(t, c + \epsilon)\\ &= f(t, c) + \epsilon\frac{\partial f}{\partial y}(t, c) + O(\epsilon^2)\\ &= \epsilon\frac{\partial f}{\partial y}(t, c) + O(\epsilon^2)\,,\end{aligned}$$

where we have performed a Taylor expansion in passing to the second line, and used $f(t, c) = 0$ in the third. Sufficiently close to $y = c$ (i.e., for ϵ suitably small), we can approximate the evolution of ϵ with

$$\frac{d\epsilon}{dt} \approx \left[\frac{\partial f}{\partial y}(t, c)\right]\epsilon\,. \tag{8}$$

This differential equation is *linear* and so is generally much simpler to solve than the original (nonlinear) equation. We can use Eq. (8) to study how the perturbation grows with time and hence determine the nature of the fixed point.

Note that if $\partial f/\partial y = 0$ at the fixed point, we must retain higher-order terms in the Taylor expansion of $f(t, c + \epsilon)$ to determine stability.

Example. For our specific example, Eq. (6), with $f(t,y) = t(1-y^2)$, we have fixed points at $y = \pm 1$ and

$$\frac{\partial f}{\partial y} = -2yt = \begin{cases} -2t & \text{at } y = 1 \ , \\ 2t & \text{at } y = -1 \ . \end{cases}$$

Therefore, near $y = 1$,

$$\frac{d\epsilon}{dt} \approx -2t\epsilon \quad \Rightarrow \quad \epsilon = \epsilon_0 e^{-t^2} ,$$

with ϵ_0 a constant. As $t \to \infty$, $\epsilon(t) \to 0$ for any ϵ_0 and so $y(t) \to 1$. It follows that $y = 1$ is a *stable* fixed point.

On the other hand, near $y = -1$,

$$\frac{d\epsilon}{dt} \approx 2t\epsilon \quad \Rightarrow \quad \epsilon = \epsilon_0 e^{t^2} .$$

Now, $|\epsilon(t)| \to \infty$ as $t \to \infty$, so if $y(t)$ starts in the vicinity of the fixed point $y = -1$, it will diverge from there.[1] It follows that $y = -1$ is an *unstable* fixed point.

4.2 Autonomous systems and phase portraits

An *autonomous* system is a special case where dy/dt is determined only by y, so that the system does not depend on time explicitly.

Definition (Autonomous system). An autonomous system is descibed by a differential equation of the form

$$\frac{dy}{dt} = f(y) , \tag{9}$$

i.e., the derivative dy/dt is only (explicitly) dependent on y.

The analysis of the stability of the fixed points is more straightforward for autonomous systems. In particular,

$\int^y \frac{du}{f(u)} = t + t_0$

If $y(t)$ is a sol'n; so is $y(t-t_0)$ $\forall t_0$.

[1] The linearised equation (8) assumes that ϵ is small and so we cannot really claim that $|\epsilon| \to \infty$ at late times. However, we can be sure that the perturbation grows as t increases.

if $y = c$ is a fixed point of Eq. (9), perturbation analysis leads to the particularly simple equation

$$\frac{d\epsilon}{dt} = \left[\frac{df}{dy}(c)\right]\epsilon = k\epsilon\,, \tag{10}$$

where k is a constant for a given fixed point. The solutions of this differential equation are of the form

$$\epsilon(t) = \epsilon_0 e^{kt}\,,$$

where ϵ_0 is a constant. It follows that the stability of the fixed point is determined by the sign of k.

Therefore, for the autonomous system (9), with fixed point $y = c$,

$$\text{if} \quad \begin{cases} f'(c) < 0 & \Rightarrow \quad \text{stable fixed point,} \\ f'(c) > 0 & \Rightarrow \quad \text{unstable fixed point.} \end{cases}$$

Example (Chemical kinetics). Consider a chemical reaction $A + B \to C + D$. Let us start with a_0 molecules of A, b_0 of B and no C nor D. Each reaction depletes the numbers of A and B molecules by one each and increases C and D similarly. We thus have:

	A	$+$	B	$\to$	C	$+$	D
Number of molecules	$a(t)$		$b(t)$		$c(t)$		$c(t)$
Intitial number of molecules	a_0		b_0		0		0

with $a(t) = a_0 - c(t)$ and $b(t) = b_0 - c(t)$.

We assume that the rate of reaction is proportional to ab (as would be appropriate for dilute gases or solutions) so that

$$\begin{aligned} \frac{dc}{dt} &= \lambda ab \\ &= \lambda(a_0 - c)(b_0 - c) \\ &= f(c)\,, \end{aligned} \tag{11}$$

where λ is a positive constant. We thus have an autonomous, first-order, nonlinear differential equation.

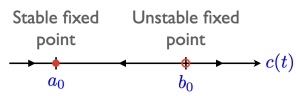
The fixed points are clearly $c = a_0$ and $c = b_0$. Let us assume that $a_0 < b_0$, in which case $c = a_0$ corresponds to having depleted all A molecules, while $c = b_0$ corresponds to the unphysical case of having depleted all B molecules (which requires $a < 0$).

We can analyse the stability by computing df/dc at the fixed points. We have

$$\frac{df}{dc} = \lambda(2c - a_0 - b_0) = \begin{cases} \lambda(a_0 - b_0) \text{ at } c = a_0 \ , \\ \lambda(b_0 - a_0) \text{ at } c = b_0 \ , \end{cases}$$

so that $c = a_0$ is a stable fixed point and $c = b_0$ is an unstable fixed point.

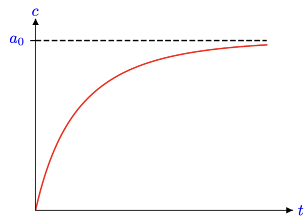
We can illustrate this behaviour with a 1D *phase portrait*, which is a plot of the dependent variable only, with arrows indicating the evolution with time. An example is shown to the right.



Finally, we note that we can easily find (exercise!) the exact solution to Eq. (11) with $c(0) = 0$:

$$c(t) = \frac{a_0 b_0 \left(1 - e^{-\lambda(b_0 - a_0)t}\right)}{b_0 - a_0 e^{-\lambda(b_0 - a_0)t}} \ .$$

This is plotted to the right.



Example (Population dynamics and the logistic equation). The logistic equation is a simple, but widely applicable, model of population dynamics. Suppose we have a population of size $y(t)$. Let the birth rate be αy, with α a positive constant. If we model the death rate as βy, with β a further positive constant, then the dynamics of the population is described by

$$\frac{dy}{dt} = (\alpha - \beta) y \, ,$$

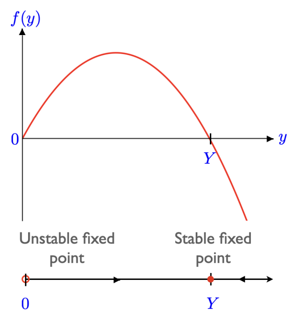
and grows or decays exponentially according to the sign of $\alpha - \beta$ (i.e., whether the birth rate exceeds the death rate or the other way around). Such a model is unrealistic, with populations often naturally regulating after early exponential growth.

We can improve the model by modifying the death rate. Suppose that to survive, members of the population must consume some resource that is limited. Let us assume that in a given time interval, the probability of a given member not finding the resource (and so dying) is proportional to y, since the rest of the population are also consuming the resource. Adding this to the death rate, we now have

$$\begin{aligned}\frac{dy}{dt} &= (\alpha - \beta)y - \gamma y^2 \\ &= \lambda y \left(1 - \frac{y}{Y}\right) , \end{aligned} \tag{12}$$

where $\lambda = \alpha - \beta$ and $Y = \lambda/\gamma$. This is the *differential logistic equation.*

The logistic equation is separable and can be easily solved exactly. However, let us reconstruct the behaviour from the phase portrait.



Equation (12) is autonomous with the derivative given by $f(y) = \lambda y(1 - y/Y)$. The fixed points are $y = 0$ and $y = Y$ and

$$\frac{df}{dy} = \lambda \left(1 - \frac{2y}{Y}\right) = \begin{cases} \lambda \text{ at } y = 0 , \\ -\lambda \text{ at } y = Y . \end{cases}$$

For $\lambda > 0$, we see that $y = 0$ is an unstable fixed point and $y = Y$ is a stable fixed point. A plot of $f(y)$ and the 1D phase portrait is shown to the right.

When the population is small ($y \ll Y$),

$$\frac{dy}{dt} \approx \lambda y$$

and there is exponential growth for $\lambda > 0$. However, as the population grows, the additional term in the death rate, $-\gamma y^2$, becomes important and the stable fixed point $y = Y$ is approached (exponentially).

4.3 Fixed points in discrete equations

We introduced discrete equations earlier as an approximation to differential equations. Now let us consider fixed points of discrete equations.

Consider a *first-order* discrete equation of the form

$$x_{n+1} = f(x_n)\,. \tag{13}$$

Definition (Fixed point of a discrete equation). A *fixed point* of a first-order discrete equation is a value of x_n such that $x_{n+1} = x_n$, i.e.,

$$f(x_n) = x_n\,.$$

We can investigate the stability of fixed points using a perturbation analysis, similiar to that used for differential equations. Suppose that x_f is a fixed point and x_n is close to x_f. If we write $x_n = x_f + \epsilon_n$, where the $\{\epsilon_n\}$ are small perturbations, the discrete equation (13) gives

$$\begin{aligned} x_f + \epsilon_{n+1} &= f(x_f + \epsilon_n) \\ &\approx f(x_f) + \epsilon_n \frac{df}{dx}(x_f) \\ \Rightarrow \quad \epsilon_{n+1} &\approx \epsilon_n \frac{df}{dx}(x_f)\,, \end{aligned}$$

where we have used $f(x_f) = x_f$ as x_f is a fixed point.

It follows that the *iterates* $\{x_n\}$ get closer to the fixed point or diverge from it according to the magnitude of df/dx at the fixed point. In particular,

$$\text{if} \quad \begin{cases} |f'(x_f)| < 1 & \Rightarrow \quad \text{stable fixed point,} \\ |f'(x_f)| > 1 & \Rightarrow \quad \text{unstable fixed point.} \end{cases}$$

Extended example (Logistic map). We illustrate these ideas with a discrete form of the differential logistic equation (12) called the *discrete logistic equation* or the *logistic map*:

$$x_{n+1} = r x_n (1 - x_n)\,. \tag{14}$$

This simple discrete equation has a remarkably rich phenomenology, some of which we shall explore in this extended example.

We can relate the logistic map to the differential logistic equation by approximating dy/dt in the latter with a finite difference in the time step Δt:

$$\frac{y_{n+1} - y_n}{\Delta t} = \lambda y_n \left(1 - \frac{y_n}{Y}\right) ,$$

so that

$$\begin{aligned} y_{n+1} &= y_n + \lambda \Delta t y_n \left(1 - \frac{y_n}{Y}\right) \\ &= (1 + \lambda\Delta t) y_n \left[1 - \left(\frac{\lambda \Delta t}{Y(1+\lambda\Delta t)}\right) y_n\right] . \end{aligned}$$

If we write

$$x_n = \left(\frac{\lambda \Delta t}{Y(1+\lambda\Delta t)}\right) y_n \quad \text{and} \quad r = (1 + \lambda \Delta t) ,$$

we recover the logistic map (14).

We are interested in non-negative iterates, $x_n > 0$. If $0 \le x_n \le 1$, the map ensures that $x_{n+1} \ge 0$ for $r > 0$. Moreover, if $r < 4$, we are ensured that $x_{n+1} \le 1$ also.

The fixed points of the logistic map satisfy $x_n = f(x_n)$, where $f(x_n) = r x_n (1 - x_n)$, and so are given by

$$x_n = 0 \quad \text{or} \quad x_n = 1 - \frac{1}{r} .$$

The fixed point at $x_n = 1 - 1/r$ is only in the physical range for $r \ge 1$.

To assess stability, we use

$$\frac{df}{dx} = r(1 - 2x) = \begin{cases} r \text{ at } x = 0 \ , \\ 2 - r \text{ at } x = 1 - 1/r \ . \end{cases}$$

We see that:

- $x_n = 0$ is a stable fixed point for $0 < r < 1$ and is unstable for $r > 1$;

- $x_n = 1 - 1/r$ is a stable fixed point for $1 < r < 3$, and is unstable for $r > 3$.

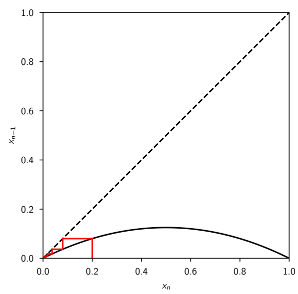
We can illustrate the evolution of the iterates using what is sometimes called a *cobweb diagram*. The idea is to plot the function $f(x)$ and the line $y = x$, so that the fixed points of $x_{n+1} = f(x_n)$ are given by the intersection of these. Starting at some initial value, say x_0, on the x-axis, we perform the following steps:

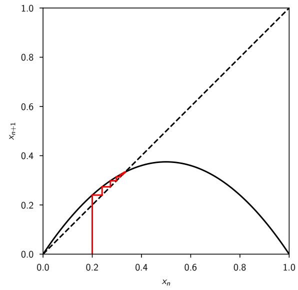
Detail's non-examinable

1. draw a vertical line from $(x_0, 0)$ to where it meets $y = f(x)$ at the point $(x_0, f(x_0))$;
2. draw a horizontal line from this point to where it meets $y = x$, so that the x value at the intersection is $f(x_0)$, i.e., x_1;
3. from this point, draw a vertical line to where it meets $y = f(x)$, followed by a horizontal line from there to the intersection with $y = x$, at which point the x value is x_2; and
4. repeat this sequence of vertical and horizontal lines as many times as required.

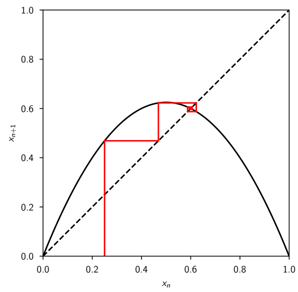
We now illustrate the behaviour of the logistic map with cobweb diagrams for different ranges of r.

$0 < r < 1$. In this case we have only a stable fixed point in the range $0 \leq x_n \leq 1$ and this is at $x_n = 0$. The iterates rapidly converge to this point, as shown in the diagram to the right (which has $r = 0.5$).

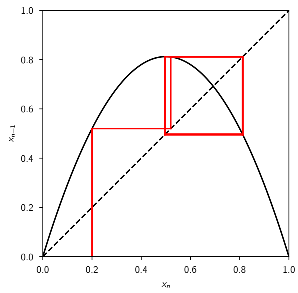




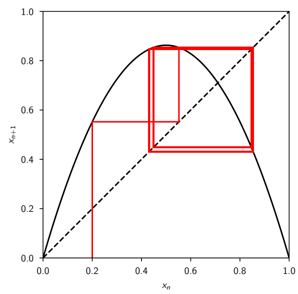
$1 < r < 2$. In this case we have an unstable fixed point, $x_n = 0$, and a stable fixed point, $x_n = 1 - 1/r$, which occurs to the left of the maximum of $f(x)$ at $x = 1/2$. Convergence to the stable fixed point is monotonic (see diagram to the right for $r = 1.5$).



$2 < r < 3$. In this case we still have an unstable fixed point, $x_n = 0$, and a stable fixed point, $x_n = 1-1/r$, but the stable fixed point occurs to the right of the maximum of $f(x)$ at $x = 1/2$. Convergence to the stable fixed point is now oscillatory (see diagram to the right for $r = 2.5$).



$3 < r < 1 + \sqrt{6}$. For $r > 3$, the two fixed points are both unstable. For $3 < r < 1 + \sqrt{6} \approx 3.44949$, for almost all starting points the iterates approach oscillations between two values on either side of the fixed point $x_n = 1 - 1/r$, so that $x_{n+2} = x_n$. This is an example of a stable *limit cycle of period 2*. The values in the limit cycle are stable fixed points of the map for the second iterates, $x_{n+2} = f[f(x_n)]$, which follows from iterating the logistic map twice. An example cobweb diagram is shown to the right for $r = 3.25$.



$1 + \sqrt{6} < r < 3.54409$. In this range, for almost all starting points the iterates oscillate between four values with $x_{n+4} = x_n$. This is an example of a stable *limit cycle of period 4*. An example is given to the right for $r = 3.451$. Note that the range of r over which the limit cycle of period 4 is attained is shorter than for the period-2 cycle.

$3.54409 < r < 3.56995$. As r moves through this range, a stable limit cycle of period 8, then 16, then 32, etc., is reached. The length of each cycle falls rapidly and the ratio of succssive intervals asymptotically approaches a

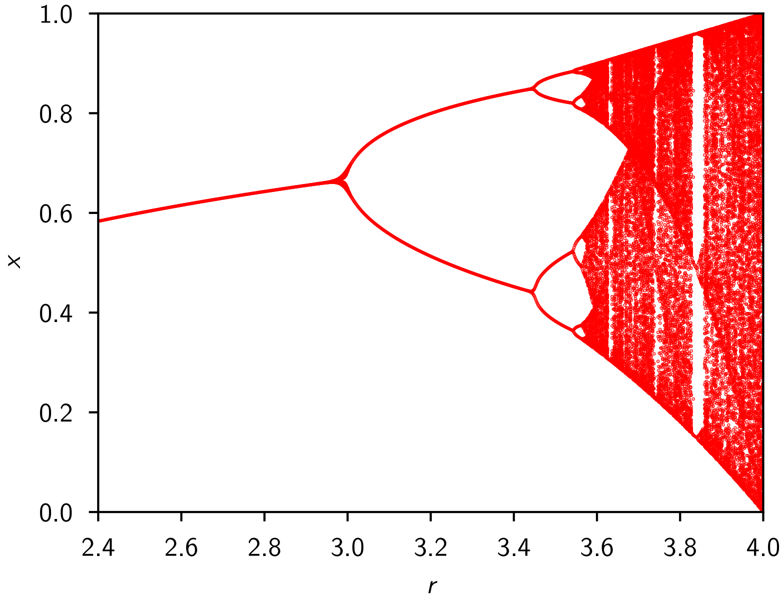
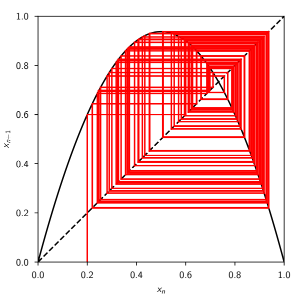


Figure 3: Bifurcation diagram for the logistic map, Eq. (14). This plots the asymptotic values of the iterates (for nearly all starting values) as r is varied. Convergence to a stable fixed point is indicated by the single branch for $r < 3$. Beyond this, the diagram bifurcates to a limit cycle of period 2, then 4, etc., of rapidly decreasing length, before chaotic behaviour ensues for $r > 3.56995$. For certain limited ranges of r beyond this, there are "islands of stability" where the map is non-chaotic.

constant (the *Feigenbaum constant*, with value approximately 4.66920). This is an example of a *period-doubling cascade*. The cascade ends at $r \approx 3.56995$.

$\boldsymbol{r > 3.56995}$. Beyond the end of the period-doubling cascade, the map becomes *chaotic*. For almost all inital conditions, the iterates no longer converge to oscillation amongst a finite number of values. Instead, the behavior is chaotic, with extreme sensitivity to the intial conditions (the example to the right has $r = 3.75$). There are, however, a few "islands of stability" – small ranges of r where the behaviour is non-chaotic and instead the iterates reach oscillation in a limit cycle.



The period-doubling cascade, the onset of chaotic behaviour and the islands of stability are illustrated in the *bifurcation diagram* in Fig. 3. This shows the asymptotic value(s) of the iterates as a function of r for nearly all starting values, with the limit cycles appearing as a finite number of branches.

IV. HIGHER-ORDER LINEAR DIFFERENTIAL EQUATIONS

We now consider linear differential equations where derivatives of the dependent variable of higher order appear, e.g., d^2y/dx^2. In particular, we shall focus on second-order equations but most of the methods in this topic apply also to higher-order equations.

1 Second-order equations with constant coefficients

We begin by considering linear, second-order differential equations with constant coefficients. These take the form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)\,, \tag{1}$$

where a, b and c are constants.

The differential operator on the left of Eq. (1) is *linear*.

Definition (Linear differential operator). A differential operator $\mathcal{D}$ is *linear* if for any $y_1(x)$ and $y_2(x)$, and constants α and β,

$$\mathcal{D}(\alpha y_1 + \beta y_2) = \alpha\mathcal{D}(y_1) + \beta\mathcal{D}(y_2)\,.$$

We can exploit linearity of $\mathcal{D}$ to solve Eq. (1) in two steps:

1. find the *complementary functions* that satisfy the homogeneous (unforced) equation, i.e.,

$$a\frac{d^2y_c}{dx^2} + b\frac{dy_c}{dx} + cy_c = 0\,;$$

2. find a *particular integral* y_p that satisfies the full equation.

A solution to the full equation can be found by adding the complementary function and particular integral since

$$\mathcal{D}(y_c + y_p) = \mathcal{D}(y_c) + \mathcal{D}(y_p) = 0 + f(x)\,.$$

If y_{c1} and y_{c2} are *linearly independent* complementary functions, then $y_{c1} + y_p$ and $y_{c2} + y_p$ are linearly independent solutions of the full equation.

Definition (Linear dependence of functions). A set of N functions $\{f_i(x)\}$ is *linearly dependent* if

$$\sum_{i=1}^{N} c_i f_i(x) = 0\,, \quad \forall x \in \text{range we are interested in}$$

for N constants $\{c_i\}$, where at least one of the c_i is nonzero. Otherwise, the functions are *linearly independent.*

Equivalently, the N functions are linearly dependent if any of them can be written as a linear combination of the others.

1.1 Complementary functions

Recall that $e^{\lambda x}$ is an eigenfunction of d/dx, i.e.,

$$\frac{d}{dx}e^{\lambda x} = \lambda e^{\lambda x}\,.$$

It follows that $e^{\lambda x}$ is also an eigenfunction of d^2/dx^2 and, indeed, of any linear differential operator with constant coefficients. Taking

$$\mathcal{D} = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c\,,$$

we have

$$\mathcal{D}\left(e^{\lambda x}\right) = \left(a\lambda^2 + b\lambda + c\right)e^{\lambda x}\,.$$

Complementary functions of Eq. (1) satisfy $\mathcal{D}(y_c) = 0$ and so are eigenfunctions with eigenvalue zero. It follows that

$$y_c = Ae^{\lambda x}$$

is a complementary function provided that the *characteristic equation* is satisfied.

Definition (Characteristic equation). The *characteristic equation* of the (second-order) differential equation $ay'' + by' + cy = 0$ is

$$a\lambda^2 + b\lambda + c = 0\,.$$

Since the characteristic equation of a second-order differential equation is quadratic, there are *two* solutions, λ_1, λ_2, leading to two complementary functions

$$y_{c1} \propto e^{\lambda_1 x} \quad \text{and} \quad y_{c2} \propto e^{\lambda_2 x}\,.$$

If $\lambda_1 \neq \lambda_2$, then y_{c1} and y_{c2} are linearly independent. Furthermore, any other solution of the homogenous differential equation $\mathcal{D}(y) = 0$ can then be written as a linear combination of y_{c1} and y_{c2}, i.e.,

$$y(x) = c_1 y_{c1}(x) + c_2 y_{c2}(x)\,.$$

This is the most general complementary function for Eq. (1), with y_{c1} and y_{c2} forming a *basis* for the space of solutions of the homogeneous equation.

You should be aware that the roots of the characteristic equation may be complex, in which case the complementary functions have oscillatory character. Moreover, the roots may be degenerate, $\lambda_1 = \lambda_2$. In this case, we only have one linearly independent complementary function of the form $e^{\lambda_1 x}$. We shall explore how to deal with this case in an example below.

Example (Non-degenerate, real roots of the characteristic equation). Consider the equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0\,.$$

The characteristic equation is

$$\lambda^2 - 5\lambda + 6 = 0 \quad \Rightarrow \quad (\lambda - 2)(\lambda - 3) = 0\,,$$

which is solved by $\lambda = 2$ or 3. It follows that the general complementary function is

$$y_c(x) = Ae^{2x} + Be^{3x}\,,$$

for arbitrary constants A and B.

Example (Complex roots). Consider the equation

$$\frac{d^2y}{dx^2} + 4y = 0\,.$$

The characteristic equation is

$$\lambda^2 + 4 = 0 \quad \Rightarrow \quad \lambda = \pm 2i\,.$$

The roots are non-degenerate but complex. The general complementary function is

$$y_c(x) = Ae^{2ix} + Be^{-2ix}\,,$$

for arbitrary (complex) constants A and B. We can express this in terms of sine and cosine to emphasise the oscillatory character:

$$\begin{aligned} y_c &= A(\cos 2x + i\sin 2x) + B(\cos 2x - i\sin 2x) \\ &= \alpha\cos 2x + \beta\sin 2x\,, \end{aligned}$$

where $\alpha = A + B$ and $\beta = i(A - B)$ are two further arbitrary constants.

Example (Degeneracy and detuning). Consider the equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0\,. \tag{2}$$

The characteristic equation is

$$\lambda^2 - 4\lambda + 4 = 0 \quad \Rightarrow \quad (\lambda - 2)^2 = 0\,.$$

The roots are now degenerate, $\lambda = 2$, and we only generate one linearly independent solution $y_c \propto e^{2x}$. This does not form a basis for the solution space of Eq. (2) since the the solution space of any second-order differential equation is two-dimensional.

We can construct a second, linearly independent solution using a technique known as *detuning*. The idea is to modify Eq. (2) slightly to remove the degeneracy. In particular, consider the "detuned" equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + \left(4 - \epsilon^2\right) y = 0 \,. \tag{3}$$

In the limit $\epsilon \to 0$, this reduces to Eq. (2), the equation that we really want to solve. The characteristic equation for Eq. (3) is

$$\lambda^2 - 4\lambda + 4 - \epsilon^2 = 0 \,,$$

which has roots $\lambda = 2 \pm \epsilon$. The general solution of the detuned equation (3) is then

$$\begin{aligned} y &= Ae^{(2+\epsilon)x} + Be^{(2-\epsilon)x} \\ &= e^{2x}\left(Ae^{\epsilon x} + Be^{-\epsilon x}\right) \,. \end{aligned}$$

To take the limit as $\epsilon \to 0$, we use the series expansion of the exponential function to obtain

$$y = e^{2x}\left[(A+B) + \epsilon(A-B)x + O(A\epsilon^2) + O(B\epsilon^2)\right] \,.$$

Suppose now that we choose to solve Eq. (2) with the initial conditions $y(0) = C$ and $y'(0) = D$. Adopting the same initial conditions for the detuned equation, we have

$$C = A + B \quad \text{and} \quad D = 2(A+B) + \epsilon(A-B) \,.$$

It follows that

$$A + B = C \quad \text{and} \quad \epsilon(A-B) = D - 2C \,.$$

Moreover, terms of $O(A\epsilon^2)$, for example, become $O(\epsilon)$ since

$$A = \frac{1}{2}\left(C + \frac{D-2C}{\epsilon}\right) \,.$$

$$= \frac{1}{2}\left(C\epsilon^2 + (D-2C)\epsilon\right)$$

This term dominates

Taking the limit as $\epsilon \to 0$, we have

$$y \to e^{2x}\left[C + (D-2C)x\right] \,.$$

It follows that the general solution of Eq. (2) is

$$y_c = e^{2x}\,(\alpha + \beta x)\ ,$$

for arbitrary constants α and β.

We see that we have constructed a second, linearly independent complementary function of the degenerate equation (2) of the form xe^{2x}. This is reminiscent of the solutions we found in the radioactivity example back in Topic II.

This example illustrates a general rule. For linear equations with constant coefficients where the characteristic equation has a repeated root, if $y_{c1}(x)$ is a degenerate complementary function, then $y_{c2}(x) = xy_{c1}(x)$ is a linearly independent complementary function.

2 Homogeneous second-order equations with non-constant coefficients

Having seen how to solve homogeneous second-order equations with constant coefficients, let us consider the more general case of equations with non-constant coefficients. In this section, we shall discuss ways to find a second, linearly independent complementary function assuming that we have been able to find a first solution. We shall also look at some general properties of these solutions.

We shall consider equations of the form

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0\,, \qquad (4)$$

for general functions $p(x)$ and $q(x)$.

2.1 Second complementary function: reduction of order

Suppose that we know a first complementary function $y_1(x)$ that solves Eq. (4).

Let us assume that the second complementary function $y_2(x) = v(x)y_1(x)$ for some as-yet-undetermined function $v(x)$. Applying the product rule, we have

$$y_2' = vy_1' + v'y_1 \quad \text{and} \quad y_2'' = vy_1'' + 2v'y_1' + v''y_1 \,.$$

Given that y_2 is supposed to satisfy Eq. (4), substituting and collecting terms we find

$$v''y_1 + v'\left(2y_1' + py_1\right) + v\left(y_1'' + py_1' + qy_1\right) = 0 \,.$$

Since y_1 also satisfies Eq. (4), the bracket multiplying $v(x)$ is zero and so

$$v''y_1 + v'\left(2y_1' + py_1\right) = 0 \,.$$

This is a *first-order* equation for the variable $u \equiv v'$:

$$u'y_1 + u\left(2y_1' + py_1\right) = 0 \,. \tag{5}$$

Equation (5) is a separable, first-order equation and so can be integrated to find $u = v'$ up to an integration constant that appears as an overall normalisation factor, A:

$$\begin{aligned} \frac{u'}{u} &= -\frac{2y_1'}{y_1} - p \\ \Rightarrow \qquad \ln u &= -2\ln y_1 - \int^x p(u)\,du + \ln A \\ \Rightarrow \qquad u(x) &= \frac{A}{y_1^2(x)} \exp\left(-\int^x p(w)\,dw\right) \,. \end{aligned} \tag{6}$$

This equation can be further integrated to determine $v(x)$ up to addition of a further integration constant. This latter constant will just lead to a y_2 containing arbitrary amounts of y_1.

This technique is called *reduction of order*, because we have reduced a second-order differential equation to a first-order equation (for v') that we can solve since it is separable. This is actually a very general technique for higher-order differential equations.

Example. Consider the degenerate equation (2):

$$y'' - 4y' + 4y = 0\,,$$

which we know has a (degenerate) solution $y_1 = e^{2x}$. We look for a second solution in the form $y_2(x) = v(x)y_1(x)$, so that

$$y_2' = (v' + 2v)e^{2x} \quad \text{and} \quad y_2'' = (v'' + 4v' + 4v)\,e^{2x}\,.$$

Substituting into the differential equation, and dividing through by e^{2x} (which is guaranteed to be non-zero for all x) we obtain

$$v'' + 4v' + 4v - 4(v' + 2v) + 4v = 0\,.$$

Unsurprisingly there is lots of cancellation, and so

$$v'' = 0 \quad \Rightarrow \quad u = v' = A\,,$$

consistent with Eq. (6) since $p(x) = -4$. Finally, we integrate again to find $v = Ax + B$ for arbitrary constants A and B. It follows that

$$y_2(x) = (Ax + B)e^{2x}\,.$$

We see that the integration constant B just adds an arbitrary amount of $y_1(x)$ to $y_2(x)$, and we can take the second, linearly independent solution to be $y_2 \propto xe^{2x}$ in agreement with that we found previously by detuning.

2.2 Phase space

Quite generally, an nth-order differential equation determines the nth derivative, $y^{(n)}(x)$ in terms of $y(x)$ and its derivatives up to $y^{(n-1)}(x)$. Moreover, by differentiating the equation, these same derivatives determine all higher-order derivatives too.

If we think of specifying $y(x_0), y'(x_0), \ldots, y^{(n-1)}(x_0)$ at some initial point x_0, then we can construct all derivatives there and hence the Taylor series of $y(x)$ about x_0.

Hence, the solution of a nth-order differential equation is uniquely determined by these initial conditions.[1]

We can therefore think of the state of the system governed by the differential equation as being fully specified at any value of the independent variable by the *solution vector* $\mathbf{Y}(x)$:

$$\mathbf{Y}(x) = \begin{pmatrix} y(x) \\ y'(x) \\ \vdots \\ y^{(n-1)}(x) \end{pmatrix} . \tag{7}$$

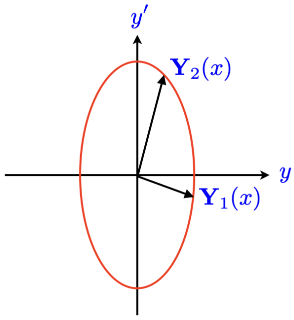
For every x, this vector (note the convention of writing it in bold font) defines a point in a n-dimensional *phase space*. As x varies $\mathbf{Y}(x)$ traces out a *trajectory* in phase space.

Example. Consider the undamped oscillator equation

$$y'' + 4y = 0 .$$

Two linearly independent solutions are $y_1 = \cos 2x$ and $y_2 = \sin 2x$. Therefore, the associated solution vectors are

$$\mathbf{Y}_1(x) = \begin{pmatrix} y_1 \\ y_1' \end{pmatrix} = \begin{pmatrix} \cos 2x \\ -2\sin 2x \end{pmatrix} ,$$
$$\mathbf{Y}_2(x) = \begin{pmatrix} y_2 \\ y_2' \end{pmatrix} = \begin{pmatrix} \sin 2x \\ 2\cos 2x \end{pmatrix} .$$



The two solution vectors thus trace out an elliptical trajectory in phase space as x varies, as shown in the figure to the right.

Note how in this example the two solution vectors are linearly independent (i.e., non-colinear in 2D) for all x. (We shall see shortly how this idea generalises.) Therefore, any point in phase space can be reached by a linear combination of $\mathbf{Y}_1$ and $\mathbf{Y}_2$ at any x and the solution vectors form a *basis* for the 2D phase space.

[1] It takes rather more work to prove this statement rigorously!

Generally, for an nth-order linear equation, it is often convenient to use the solution vectors constructed from n linearly independent complementary functions as basis vectors for the phase space.

2.3 Wronskian and linear independence

Recall that n functions, $y_i(x)$ $(i = 1, \ldots, n)$, are linearly *dependent* if

$$\sum_{i=1}^{n} c_i y_i(x) = 0$$

for some n constants c_i not all of which are zero. Since this has to hold for all x (within some domain of interest), we can differentiate $n-1$ times and collect the n constraints as

$$\sum_{i=1}^{n} c_i \mathbf{Y}_i(x) = 0\,, \tag{8}$$

where the $\mathbf{Y}_i(x)$ are the n vectors constructed from the y_i and their derivatives. This vector equation is the statement that the n vectors $\mathbf{Y}_i(x)$ are linearly dependent for all x.

Equation (8) implies that the determinant of the *fundamental matrix*, constructed with the ith column being $\mathbf{Y}_i(x)$, vanishes if the functions $y_i(x)$ $(i = 1, \ldots, n)$ are linearly dependent.[2] We call this determinant the *Wronskian*.

Definition (Wronskian). The *Wronskian* $W(x)$ of n functions $y_i(x)$ $(i = 1, \ldots, n)$ is the determinant of the fundamental matrix whose columns are the vectors $\mathbf{Y}_i$:

$$W(x) \equiv \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix} .$$

[2] If the vectors are linearly dependent, some column of the fundamental matrix is a linear combination of other columns and so the determinant vanishes.

We have seen that

$$\text{linear dependence of the } y_i(x) \quad \Rightarrow \quad W(x) = 0\,.$$

It follows that[3]

$$W(x) \neq 0 \quad \Rightarrow \quad \text{the } y_i(x) \text{ are linearly independent.}$$

This is very useful in the context of solutions of nth-order linear differential equations as we can test for linear independence by calculating the Wronskian of n putative solutions.

For a second-order differential equation, as considered here, the Wronskian $W(x)$ takes the particularly simple form

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'. \qquad (9)$$

Example. For the equation $y'' + 4y = 0$, we have $y_1 = \cos 2x$ and $y_2 = \sin 2x$. Therefore

$$W(x) = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2(\cos^2 2x + \sin^2 2x) = 2\,,$$

so these solutions are linearly independent (as expected).

Example. For the equation $y'' - 4y' + 4y = 0$, we have $y_1 = e^{2x}$ and the independent solution $y_2 = xe^{2x}$. In this case,

$$W(x) = \begin{vmatrix} e^{2x} & xe^{2x} \\ e^{2x} & (1+2x)e^{2x} \end{vmatrix} = e^{4x}(1 + 2x - 2x) = e^{4x}\,,$$

establishing linear independence again as e^{4x} is never zero.

[3]Note that $W = 0$ does *not* necessarily imply linear *dependence*. For example, in two dimensions sufficient conditions for linear dependence are that $W = 0$ *and* that one of the functions, say y_1, is non-zero over the domain of interest. In this case, we can write the Wronskian in the form

$$W(x) = -y_1^2 \frac{d}{dx}\left(\frac{y_2}{y_1}\right)$$

and then $W = 0$ implies $y_2 \propto y_1$.

2.4 Abel's theorem

In these examples, the Wronskian is non-zero for all x. Is it possible that it is zero for some x while non-zero for others? The answer is no, as a consequence of *Abel's theorem.*

Theorem (Abel's theorem). Given a linear, second-order, homogenous differential equation,

$$y'' + p(x)y' + q(x) = 0\,, \tag{10}$$

where $p(x)$ and $q(x)$ are continuous on an interval I, then, for any solutions of the differential equation, either $W = 0$ for all $x \in I$ or $W \neq 0$ for all $x \in I$.

Proof (sketch). From Eq. (9), the derivative of the Wronskian of two solutions of Eq. (10) is

$$W' = y_1 y_2'' - y_2 y_1''\,.$$

Since y_1 and y_2 satisfy the differential equation (10), we have

$$\begin{aligned} W' &= y_2\left(p y_1' + q y_1\right) - y_1\left(p y_2' + q y_2\right) \\ &= -p\left(y_1 y_2' - y_2 y_1'\right) \\ &= -pW\,. \end{aligned}$$

This is a separable, first-order equation for the Wronskian with solution (*Abel's identity*)

$$W(x) = W(x_0)\exp\left(-\int_{x_0}^{x} p(u)\,du\right)\,, \tag{11}$$

for some arbitrary x_0. Since the exponential is never zero, either $W(x_0) = 0$, in which case $W = 0$ for all x, or $W(x_0) \neq 0$, in which case $W \neq 0$ for any x.

As a corollary of Abel's identity, note that if $p(x) = 0$, i.e., the differential equation has no y' term, the Wronskian is constant.

Example (Bessel's equation). Consider

$$x^2 y'' + x y' + \left(x^2 - n^2\right) y = 0\,.$$

This equation has no closed-form solutions for most values of n ((half-integer values are an exception). Dividing through by x^2, we have

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right) y = 0\,,$$

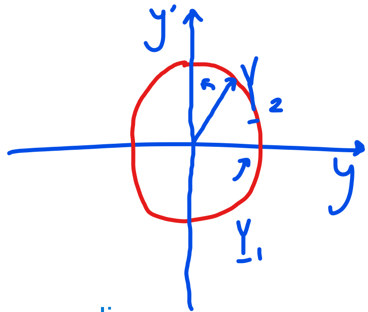
Geometric Interpretation

so that $p(x) = 1/x$. Abel's identity (11) gives

$$W(x) = W(x_0)\exp\left(-\int_{x_0}^{x} \frac{du}{u}\right)$$

$$\Rightarrow \quad W(x) = W(x_0)\frac{x_0}{x}\,.$$

Phase space



Note how Abel's identity determines the form of the Wronskian without having to solve the differential equation directly.

Solution vectors always colinear or never colinear as x varies, basis at some x $\Rightarrow$ basis $\forall x$.

2.4.1 Application of Abel's theorem

Abel's identity (11) can be written as

$$y_1y_2' - y_2y_1' = W(x_0)\exp\left(-\int_{x_0}^{x} p(u)\,du\right)\,.$$

If y_1 is known, we can think of this is a first-order equation for y_2, which we can write as

$$\frac{d}{dx}\left(\frac{y_2}{y_1}\right) = \frac{W(x_0)}{y_1^2(x)}\exp\left(-\int_{x_0}^{x} p(u)\,du\right)\,. \qquad (12)$$

Integrating both sides, we obtain $y_2(x)$ up to the addition of a constant multiple of $y_1(x)$ (which arises from the integration constant). The $y_2(x)$ that we obtain is ensured to have the correct Wronskian with $y_1(x)$, and in particular the value $W(x_0)$ at $x = x_0$. (Note that adding any multiple of $y_1(x)$ to $y_2(x)$ does not change the Wronskian.)

This method of finding a second solution to a homogeneous equation is equivalent to the method of reduction of order (Sec. 2.1). Indeed, Eq. (12) is exactly Eq. (6) on recalling that $y_2 = v(x)y_1$ and $u = v'$ there.

2.4.2 Generalisation

Abel's theorem generalises to the Wronskian of solutions of higher-order, linear, homogeneous equations (see Question 7 on Examples Sheet 3).

As we discuss further in Topic V, any nth-order, linear, homogeneous differential equation for $y(x)$ can be written in the form

$$\mathbf{Y}' + \mathbf{A}(x)\mathbf{Y} = \mathbf{0}\,,$$

where $\mathbf{A}(x)$ is an $n \times n$ matrix, which may depend on x, and $\mathbf{Y}$ is the n-dimensional vector formed from $y(x)$ and its first $n-1$ derivatives (see Eq. 7). For example, for the equation

$$y''' + p(x)y'' + q(x)y' + r(x)y = 0\,,$$

we have

$$\mathbf{Y}' + \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ r(x) & q(x) & p(x) \end{pmatrix} \mathbf{Y} = \mathbf{0}\,.$$

It can be shown that the Wronskian W of n solutions of the original nth-order differential equation satisfies

$$W' + \mathrm{Tr}[\mathbf{A}(x)]W = 0\,, \qquad \text{(qn7, sheet 3)}$$

where Tr denotes the trace. This equation is solved by

$$W = W(x_0)\exp\left(-\int_{x_0}^{x} \mathrm{Tr}[\mathbf{A}(u)]\,du\right)\,,$$

and so Abel's theorem still holds.

2.5 Linear equidimensional equations

Definition (Equidimensional equation). A linear, second-order equation is *equidimensional* if it has the form

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = f(x)\,, \tag{13}$$

where a, b and c are constants.

Such equations are called "equidimensional" (or sometimes "homogeneous", although this is confusing as we are using the latter term for equations with no forcing term, i.e., $f(x) = 0$) since solutions of the unforced equation remain solutions under scaling of x. Specifically, let $y(x)$ be a solution of Eq. (13) when $f(x) = 0$. Now consider the new function $\phi(x) = y(\alpha x)$, where α is an arbitrary scaling parameter. Since, by the chain rule,

$$x\frac{d\phi}{dx} = (\alpha x)y'(\alpha x)\,,$$

we see that

$$ax^2\frac{d^2\phi}{dx^2} + bx\frac{d\phi}{dX} + c\phi = 0\,,$$

so $\phi(x)$ satisfies the same equation as $y(x)$.

Equivalently, if Eq. (13) describes some physical system, so that the variables have dimensions, the dimensions of each term on the left-hand side are the same (assuming that the constants a, b and c to be dimensionless). For example, in a dynamical system y might have dimensions of length (L) and x dimensions of time (T). Then y' has dimensions LT^{-1} and y'' has dimensions LT^{-2}, so that y, xy and xy'' all have dimensions L.

Let us now determine the complementary functions of Eq. (13).

2.5.1 Solving by eigenfunctions

Noting that $y = x^k$ is an eigenfunction of xd/dx (with eigenvalue k), we can look for complementary functions of this form. Substituting into

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0\,,$$

we require

$$ak(k-1) + bk + c = ak^2 + (b-a)k + c = 0\,. \qquad (14)$$

This equation has two roots k_1 and k_2, and so, provided they are distinct, the general complementary function is

$$y_c = Ax^{k_1} + Bx^{k_2}\,.$$

2.5.2 Solving by substitution

An alternative method for determing the complementary functions of equidimensional equations is to make the substitution $z \equiv \ln x$. This is useful since it turns Eq. (13) into an equation with constant coefficients.

To see this, note that for $y(e^z)$,

$$\begin{aligned} \frac{dy}{dz} &= e^z y'(e^z)\,, \quad \to xy'(x) \\ \frac{d^2y}{dz^2} &= e^z y'(e^z) + e^{2z} y''(e^z)\,, \quad \to x^2 y''(x) \end{aligned}$$

so that if $y(x)$ is a solution of Eq. (13), then $y(e^z)$ satisifes

$$a\frac{d^2y}{dz^2} + (b-a)\frac{dy}{dz} + cy = f(e^z)\,. \tag{15}$$

We can now use the techniques for equations with constant coefficients (Sec. 1) to solve Eq. (15). For example, we can look for complementary functions of the form $y = e^{\lambda z}$, in which case we require

$$a\lambda^2 + (b-a)\lambda + c = 0\,.$$

This is the same characteristic equation as Eq. (14), so the roots are k_1 and k_2. The general complementary function is then

$$y_c = Ae^{k_1 z} + Be^{k_2 z} = Ax^{k_1} + Bx^{k_2}\,,$$

as expected.

In the degenerate case, the roots of the characteristic equation are equal: $\lambda = k_1 = k_2 = k$. However, we know from our earlier example of "detuning" how to deal with such cases for equations with constant coefficients. The general complementary function is then

$$y_c = Ae^{kz} + Bze^{kz} = Ax^k + Bx^k \ln x\,.$$

$f(x)$	$y_p(x)$
e^{mx}	Ae^{mx}
$\sin kx$ or $\cos kx$	$A\sin kx + B\cos kx$
Polynomial $p_n(x)$	$q_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$

Table 1: Form of particular integrals $y_p(x)$ for linear, second-order equations with constant coefficients with some common forcing terms $f(x)$.

3 Inhomogeneous (forced) second-order differential equations

So far, we have focused on finding complementary functions. Let us now consider determining particular integrals of inhomogeneous equations, beginning with the case of equations with constant coefficients.

3.1 Particular integrals of equations with constant coefficients

We consider linear, second-order equations with constant coefficients, i.e.,

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)\,.$$

For particularly simple forms of the *forcing function* $f(x)$ we can write down a particular integral $y_p(x)$ by inspection/guesswork. Table 1 lists some common cases.

The various arbitrary constants in the particular integrals are determined by substitution in the underlying differential equation. It is important to remember that this equation is *linear*, so we can superpose solutions and consider each forcing term separately.

Example. Consider the equation

$$y'' - 5y' + 6y = 2x + e^{4x}\,.$$

For the forcing term $2x$ we consider a particular integral $ax + b$, and for e^{4x} we consider ce^{4x}. Superposing, we try

$$y_p = ax + b + ce^{4x}\,,$$

so that

$$y_p' = a + 4ce^{4x}\,, \quad y_p'' = 16ce^{4x}\,.$$

Substituting these into the differential equation, and collecting terms multiplying the same function of x, we obtain

$$\underbrace{(16c - 20c + 6c)}_{\to 1} e^{4x} + \underbrace{(6a)}_{\to 2} x + \underbrace{(-5a + 6b)}_{\to 0} = 2x + e^{4x}\,.$$

Comparing coefficients, we find that $c = 1/2$, $a = 1/3$ and $b = 5/18$. Noting that the homogeneous equation is solved by e^{2x} and e^{3x}, we have the general solution

$$y = Ae^{3x} + Be^{2x} + \frac{1}{2}e^{4x} + \frac{1}{3}x + \frac{5}{18}\,.$$

Note that the boundary conditions used to determine the constant A and B must be applied to the entire solution, not just the complementary function.

3.1.1 Resonance

In the example above, the forcing term e^{4x} is not a complementary function of the differential equation. However, if the forcing term were e^{2x}, say, we would not have been able to construct a particular integral of the form $y_p(x) \propto e^{2x}$. We can deal with such cases by "detuning" the forcing term.

The process is best illustrated with a concrete example. Consider the forced equation

$$\ddot{y} + \omega_0^2 y = \sin \omega_0 t\,. \tag{16}$$

The complementary function is

$$y_c(t) = A \sin \omega_0 t + B \cos \omega_0 t\,,$$

where A and B are constants. Physically, this is an example of a simple harmonic oscillator, with natural frequency ω_0, being driven by a oscillatory force that is at the natural frequency. In such situations, the system is said to be driven *resonantly*. Since the forcing term is a complementary function at resonance, a linear combination of $\sin\omega_0 t$ and $\cos\omega_0 t$ is *not* a particular integral.

Instead, we proceed by "detuning" the forcing term away from the natural frequency by considering

$$\ddot{y} + \omega_0^2 y = \sin\omega t \quad (\omega \neq \omega_0)\,.$$

We try a particular integral

$$y_p(t) = \sin\omega t\,,$$

where C is a constant to be determined. The form of the left-hand side of Eq. (3.1.1) means there is no $\cos\omega t$ term in y_p. Substituting into the differential equation we find

$$\ddot{y}_p + \omega_0^2 y_p = C\left(-\omega^2 + \omega_0^2\right)\sin\omega t \quad \Rightarrow \quad C = \frac{1}{\omega_0^2 - \omega^2}\,.$$

Now we recall that, ultimately, we are interested in the limit $\omega \to \omega_0$. The particular integral we have found does not have a finite limit as $C \to \infty$ there. However, we can try and fix this by adding in any solution of the homogeneous equation as this will still be a valid particular integral. If we take

$$y_p(t) = \frac{1}{\omega_0^2 - \omega^2}\left(\sin\omega t - \sin\omega_0 t\right)\,, \tag{17}$$

evaluating the (now-indeterminate) limit with l'Hôpital's rule, we have

$$\lim_{\omega\to\omega_0} y_p(t) = -\frac{t}{2\omega_0}\cos\omega_0 t\,. \tag{18}$$

This is a valid particular integral of Eq. (16).

As a general rule, if the forcing term is a linear combination of complementary functions, then the particular integral is proportional to the independent variable (t

in the example above) times a non-resonant particular integral ($\cos\omega_0 t$ above). (just the guess from table 1)

Aside: Behaviour close to resonance

It is interesting to consider the particular integral in Eq. (17) in the case that the driving frequency ω is close to, but not equal to, the natural frequency ω_0.

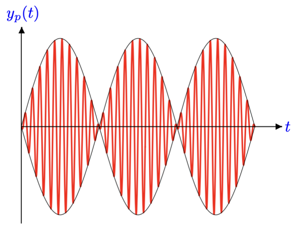
In this case, it is convenient to factorise $y_p(t)$ using

$$\begin{aligned}\sin\omega t - \sin\omega_0 t &= \operatorname{Im}\left(e^{i\omega t} - e^{i\omega_0 t}\right)\\ &= \operatorname{Im}\left[e^{i(\omega+\omega_0)t/2}\left(e^{i(\omega-\omega_0)t/2} - e^{-i(\omega-\omega_0)t/2}\right)\right]\\ &= 2\cos\left[\left(\frac{\omega+\omega_0}{2}\right)t\right]\sin\left[\left(\frac{\omega-\omega_0}{2}\right)t\right].\end{aligned}$$

This factors out an oscillation at the average frequency, $(\omega+\omega_0)/2$, and a slow oscillation at (half) the difference of the frequencies, $(\omega-\omega_0)/2$. If we write $\omega_0 - \omega = \Delta\omega$, we have

$$y_p(t) = \frac{-2}{(2\omega+\Delta\omega)\Delta\omega}\cos\left[\left(\omega+\frac{\Delta\omega}{2}\right)t\right]\sin\left(\frac{\Delta\omega}{2}t\right). \qquad (19)$$

For $\Delta\omega \ll \omega$, the slow oscillation at frequency $\Delta\omega/2$ is much less rapid than that at the average frequency. In this limit, we observe the phenomenon of *beating*: the slow oscillation acts as a long-period modulation of the amplitude of the rapid oscillation. An example is shown in the figure to the right.



As $\Delta\omega \to 0$, the period of the modulation tends to infinity and we have linear growth of the amplitude of the oscillation at $\omega = \omega_0$. It is straightforward to show that in this limit, Eq. (19) reduces to our earlier result (18).

3.1.2 Resonance in equidimensional equations

The discussion above of resonance in equations with constant coefficients carries over to the case of equidimensional equations (Sec. 2.5). Recall that such equations have complementary functions of the $y_c \propto x^{k_1}$ or x^{k_2} (in the non-degenerate case). In the case that the forcing term is proportional to x^{k_1} (or x^{k_2}), there is a particular integral of the form $x^{k_1} \ln x$.

This result follows from transforming the equidimensional equation to one with constant coefficients by the substitution $z = \ln x$. For the transformed equation, a forcing term proportional to $e^{k_1 z}$ (or $e^{k_2 z}$) gives rise to a complementary function of the form $ze^{k_1 z}$ or, expressed in terms of x, $y_p(x) \propto x^{k_1} \ln x$.

3.2 Variation of parameters

So far, we have been determining particular integrals by making an educated guess. The method of *variation of parameters* provides a systematic way to find a particular integral given linearly independent complementary functions $y_1(x)$ and $y_2(x)$.

Consider the forced (inhomogeneous) second-order differential equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = f(x)\,, \tag{20}$$

with linearly independent complementary functions y_1 and y_2.

Recall that for any solution, $y(x)$, of Eq. (20), we define the solution vector

$$\mathbf{Y}(x) = \begin{pmatrix} y(x) \\ y'(x) \end{pmatrix}. \tag{21}$$

It will prove convenient to use the vectors

$$\mathbf{Y}_1(x) = \begin{pmatrix} y_1(x) \\ y_1'(x) \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(x) = \begin{pmatrix} y_2(x) \\ y_2'(x) \end{pmatrix}$$

as a basis for the solution space, so that at any x we write a particular integral as

$$\mathbf{Y}_p(x) = u(x)\mathbf{Y}_1(x) + v(x)\mathbf{Y}_2(x)\,, \tag{22}$$

where $u(x)$ and $v(x)$ are functions to be determined.

Note that linear independence of the functions y_1 and y_2 ensures that the vectors $\mathbf{Y}_1$ and $\mathbf{Y}_2(x)$ are also linearly independent for *all* x.

The components of Eq. (22) give

$$y_p = uy_1 + vy_2\,, \tag{23}$$

$$y_p' = uy_1' + vy_2'\,. \tag{24}$$

Differentiating the second equation, we have

$$y_p'' = uy_1'' + u'y_1' + vy_2'' + v'y_2'\,. \tag{25}$$

Adding this to $p(x)$ times Eq. (24) and $q(x)$ times Eq. (23), and demanding that y_p satisfies the differential equation (20), we have

$$u'y_1' + v'y_2' = f\,, \tag{26}$$

where we have used that y_1 and y_2 are complementary functions.

Now note that we derived Eq. (24) from the second row of the vector equation (22). However, this expression for y_p' has to be consistent with what we get by differentiating y_p in Eq. (23) directly. This requires that

$$u'y_1 + v'y_2 = 0\,.$$

Along with Eq. (26), this gives us two simultaneous equations for u' and v', which we should be able to solve.

Writing these simultaneous equations in matrix form, we have

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}.$$

Inverting the matrix on the left, we have

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \frac{1}{W}\begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix}\begin{pmatrix} 0 \\ f \end{pmatrix},$$

where $W(x)$ is the (non-zero) Wronskian of y_1 and y_2. It follows that

$$u' = -\frac{y_2}{W}f \quad \text{and} \quad v' = \frac{y_1}{W}f\,. \tag{27}$$

Integrating these, and then substituting in Eq. (23), we obtain a particular integral

$$y_p(x) = y_2(x)\int^x \frac{y_1(\xi)f(\xi)}{W(\xi)}\,d\xi - y_1(x)\int^x \frac{y_2(\xi)f(\xi)}{W(\xi)}\,d\xi\,.$$

Note that we have not specified the lower limits on the integrals here. Changing these just adds constant multiples of y_1 and y_2 (i.e., a complementary function) to the particular integral.

Example. Consider the differential equation

$$y'' + 4y = \sin 2x\,.$$

Since the complementary functions are

$$y_1 = \sin 2x \quad \text{and} \quad y_2 = \cos 2x\,,$$

this is an example of an oscillator being driven resonantly (the forcing term is a complementary function). The Wronskian of y_1 and y_2 is $W = -2$, and so Eq. (27) gives

$$\begin{aligned} u' &= \frac{1}{2}\cos 2x \sin 2x = \frac{1}{4}\sin 4x\,, \\ v' &= -\frac{1}{2}\sin^2 2x = \frac{1}{4}\left(\cos 4x - 1\right)\,. \end{aligned}$$

Integrating gives

$$u = -\frac{1}{16}\cos 4x \quad \text{and} \quad v = \frac{1}{16}\sin 4x - \frac{x}{4}\,.$$

It follows that

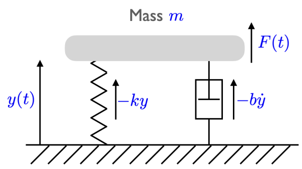
$$\begin{aligned} y_p &= \frac{1}{16}\left(-\cos 4x \sin 2x + \sin 4x \cos 2x\right) - \frac{1}{4}x\cos 2x\,, \\ &= \frac{1}{16}\sin 2x\left(-\cos 4x + 2\cos^2 2x\right) - \frac{1}{4}x\cos 2x\,, \\ &= \frac{1}{16}\sin 2x - \frac{1}{4}x\cos 2x\,. \end{aligned}$$

The first term is clearly a multiple of one of the complementary functions. The second term is of the form x times a complementary function, as expected from our earlier discussion of resonance. Indeed, identifying $\omega_0 = 2$ and t with x in Eq. (16), we see that the term $-(x\cos 2x)/4$ above is exactly the same as the particular integral (18) that we found using detuning.

4 Forced oscillating systems: transients and damping

In this section we consider linear systems where there is a *restoring force* that tends to make the system oscillate and some *damping force* (e.g., friction in a mechanical system) that tends to oppose motion. There may also be some driving force.

Consider the set-up in the figure to the right, which might describe a car suspension system, for example. A mass m is acted on by a linear restoring force $-ky$, where k is a (positive) spring constant and $y(t)$ is the vertical displacement of the mass from its equilibrium position. There is also a damping force $-b\dot{y}$, where b is a positive damping constant, that resists the motion. In addition, the system is driven by an external force $F(t)$.



The equation of motion for the displacement of the mass is given by Newton's second law: $m\ddot{y} =$ total force. We have

$$m\ddot{y} = -ky - b\dot{y} + F(t)$$
$$\Rightarrow \quad m\ddot{y} + b\dot{y} + ky = F(t)\,. \tag{28}$$

In the absence of damping and the external driving force, the system undergoes simple-harmonic motion with angular frequency $\omega_0 \equiv \sqrt{k/m}$. It is convenient to introduce a dimensionless time coordinate $\tau \equiv \omega_0 t$. Dividing Eq. (28) through by $m\omega_0^2$ (i.e., k), we can put the equation in dimensionless form:

tau is the period of oscillation up to a factor of 2pi.

$$y'' + 2\kappa y' + y = f(\tau)\,, \tag{29}$$

where

$$y' \equiv \frac{dy}{d\tau}\,, \quad \kappa = \frac{b}{m\omega_0} \quad f \equiv \frac{F}{k}\,.$$

In the form of Eq. (29), the unforced system is described by a single dimensionless parameter κ.

4.1 Free (unforced or natural) response

With $f = 0$, Eq. (29) reduces to

$$y'' + 2\kappa y' + y = 0\,. \qquad (30)$$

We look for solutions of the form $y \propto e^{\lambda\tau}$, which gives the characteristic equation

$$\lambda^2 + 2\kappa\lambda + 1 = 0\,.$$

There are (generally) two roots given by

$$\lambda_1\,, \lambda_2 = -\kappa \pm \sqrt{\kappa^2 - 1}\,.$$

There are three cases to consider: $\kappa < 1$, $\kappa = 1$ and $\kappa > 1$.

4.1.1 Light damping (underdamping): $\kappa < 1$

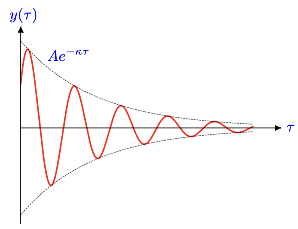
If $\kappa < 1$, both roots λ_1 and λ_2 are complex. We can write them as

$$\lambda_1\,, \lambda_2 = -\kappa \pm i\sqrt{1-\kappa^2}\,.$$

The general solution of Eq. (30) is then

$$y(\tau) = e^{-\kappa\tau}\left[A\sin\left(\sqrt{1-\kappa^2}\tau\right) + B\cos\left(\sqrt{1-\kappa^2}\tau\right)\right]\,,$$

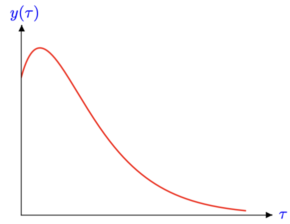
where A and B are constants. This solution describes damped oscillations at angular frequency



$$\omega_{\text{free}} = \sqrt{1-\kappa^2}\omega_0\,, \qquad (31)$$

which is lowered by damping from ω_0. The amplitude of the oscillation decays in time with a characteristic (dimensionless) decay time of $1/\kappa$. This behaviour is illustrated in the figure to the right.

4.1.2 Critical damping: $\kappa = 1$



For $\kappa = 1$, the characteristic equation is degenerate with a repeated root

$$\lambda_1 = \lambda_2 = -\kappa\,.$$

The general solution of Eq. (30) is then

$$y(\tau) = e^{-\kappa\tau}\left(A + B\tau\right)\,,$$

where A and B are constants. An example behaviour is shown in the figure to the right.

4.1.3 Heavy damping (overdamping): $\kappa > 1$

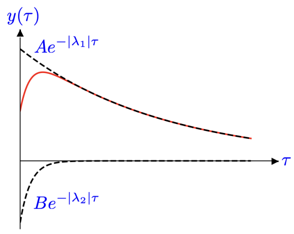
If $\kappa > 1$, both roots of the characteristic equation are real and negative. If we take $|\lambda_1| < |\lambda_2|$, then

$$\lambda_1 = -\kappa + \sqrt{\kappa^2 - 1} \quad \text{and} \quad \lambda_2 = -\kappa - \sqrt{\kappa^2 - 1}\,,$$

and the general solution of Eq. (30) is

$$y(\tau) = Ae^{-|\lambda_1|\tau} + Be^{-|\lambda_2|\tau}\,,$$

where A and B are constants.



The solution initially varies on the more rapid timescale $1/|\lambda_2|$, but once this has decayed the solution approaches $y(\tau) = Ae^{-|\lambda_1|\tau}$, which has decay time $1/|\lambda_1|$. The typical behaviour is shown in the figure to the right.

Note that in all cases, the unforced response decays eventually.

4.2 Forced response

In a forced system described by Eq. (29), the complementary function (i.e., unforced response) decays in time. The long-term behaviour is therefore determined by the driving force (through the particular integral), while the initial transient response is determined by *both* the complementary function and particular integral.

Example. Consider sinusoidal forcing so that Eq. (28) can be written as

$$\ddot{y} + \mu\dot{y} + \omega_0^2 y = \frac{F_0}{m}\sin\omega t\,, \tag{32}$$

where $\mu \equiv b/m$. We shall assume light damping, $\mu < 2\omega_0$, in which case the complementary function is

$$y_c(t) = e^{-\mu t/2}\left(A\sin\omega_{\text{free}}t + B\cos\omega_{\text{free}}t\right)\,,$$

where $\omega_{\text{free}} = \sqrt{\omega_0^2 - \mu^2/4}$.

For the particular integral, we try

$$y_p(t) = \frac{F_0}{m}\left(C\sin\omega t + D\cos\omega t\right)\,.$$

Substituting in Eq. (32) and comparing coefficients of the $\sin\omega t$ and $\cos\omega t$ terms, we have

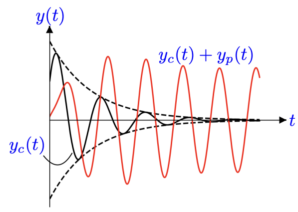
$$-\omega^2 C - \mu\omega D + \omega_0^2 C = 1$$
$$-\omega^2 D + \mu\omega C + \omega_0^2 D = 0\,.$$

These are solved by

$$C = \frac{\omega_0^2 - \omega^2}{\left(\omega_0^2 - \omega^2\right)^2 + (\mu\omega)^2}$$
$$D = \frac{-\mu\omega}{\left(\omega_0^2 - \omega^2\right)^2 + (\mu\omega)^2}$$

The full solution is $y(t) = y_c(t) + y_p(t)$, with

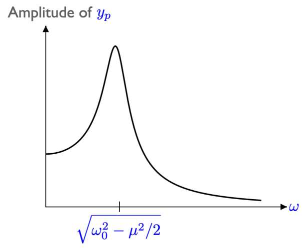
$$y_p(t) = \frac{F_0/m}{\left(\omega_0^2 - \omega^2\right)^2 + (\mu\omega)^2} \times \left[\left(\omega_0^2 - \omega^2\right)\sin\omega t - \mu\omega\cos\omega t\right]\,.$$



The solution is shown to the right. Note that the complementary function decays leaving only the particular integral asymptotically. This means that the damped oscillator has no long-term memory of its initial conditions since these only affect the constants A and B in the complementary function.

The particular integral determines the steady-state response to the driving force. This is sinusoidal with amplitude given by $(F_0/m)\sqrt{C^2+D^2}$:

$$\text{Amplitude of } y_p = \frac{F_0/m}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+(\mu\omega)^2}}\,.$$



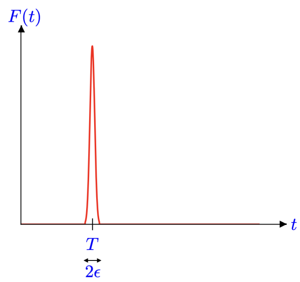
For light damping, the amplitude is sharply peaked at the *amplitude resonant frequency*

$$\omega_{\text{res}} = \sqrt{\omega_0^2-\mu^2/2}\,,$$

as illustrated in the figure to the right. The amplitude at ω_{res} increases as the damping is reduced and the peak becomes sharper.

5 Impulses and point forces

Consider a system that experiences a sudden force, extending from time $t = T-\epsilon$ to $t = T+\epsilon$, where ϵ is small compared to any other natural timescale (e.g., oscillation period or decay time) in the system. For example, the damped oscillator discussed in Sec. 4 might be struck from being at rest at its equilibrium position at time T (see figure to the right).



For small enough ϵ, the subsequent behaviour of the system does not depend on ϵ or the detailed form of the force when non-zero – all that matters is the *impulse* of the force,

$$I \equiv \int_{T-\epsilon}^{T+\epsilon} F(t)\,dt\,.$$

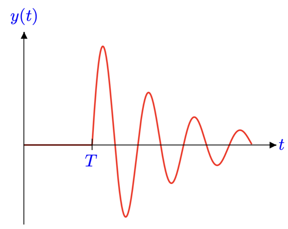
Mathematically, it is then convenient to consider the limit of a sudden, impulsive force with $\epsilon \to 0$, while preserving the impulse. (This means that the magnitude of the force at its peak must grow without limit.)

If we consider the damped, driven oscillator of Sec. 4, described by

$$m\ddot{y} + b\dot{y} + ky = F(t)\,,$$

we can integrate the equation of motion from $T - \epsilon$ to $T + \epsilon$ to find

$$\lim_{\epsilon\to 0}\left(m\,[\dot{y}]_{T-\epsilon}^{T+\epsilon} + b\,[y]_{T-\epsilon}^{T+\epsilon} + k\int_{T-\epsilon}^{T+\epsilon} y\,dt\right) = I\,.$$



The second term on the left is zero if y is continuous and the third is zero if y is finite in the interval. If we assume these to be true (they certainly will be in any physical system), we see that the velocity $\dot{y}$ undergoes a sudden change (it is discontinuous) that depends on the impulse of the force:

$$\lim_{\epsilon\to 0}\left(m\,[\dot{y}]_{T-\epsilon}^{T+\epsilon}\right) = I\,.$$

A typical behaviour is shown in the figure to the right.

5.1 The Dirac delta function

We can formalise the idea of an impulsive force by introducing the *Dirac delta function*.

We consider a family of functions $D(t;\epsilon)$ that have two key properties:

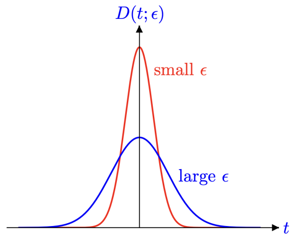
1. $\lim_{\epsilon\to 0} D(t;\epsilon) = 0\ \forall\ t \neq 0\,;$
2. $\int_{-\infty}^{\infty} D(t;\epsilon)\,dt = 1\,.$

The sudden force in the example above can then be represented by $F(t) = I D(t - T;\epsilon)$.

Multiply by I, to get correct impulse, and shift by T

An example of such a family is

$$D(t;\epsilon) = \frac{1}{\epsilon\sqrt{\pi}} e^{-t^2/\epsilon^2} \,,$$



which is illustrated in the figure to the right. (These functions do integrate to unity; see Question 14 on Examples Sheet 1.) Note that as ϵ decreases, the standard deviation or "width" of the Gaussian gets narrower while the peak value gets larger, preserving the integral. Therefore, $\lim_{\epsilon\to 0} D(0;\epsilon)$ does not exist.

Of course, the family of $D(t;\epsilon)$ having these two defining characteristics is not unique. However, for any such family the limit as $\epsilon \to 0$ yields a function (more carefully, a *distribution*), which we call the *Dirac delta function*.

Definition (Dirac delta function). The *Dirac delta function* is defined by

$$\delta(x) \equiv \lim_{\epsilon\to 0} D(x;\epsilon) \,,$$

on the understanding that we can only use its integral properties, i.e., when the delta function is multiplied by some suitably well-behaved "test function" and integrated over an appropriate interval (see below).

The delta function satisfies three key properties:

1. $\delta(x) = 0 \;\forall\; x \neq 0\,;$
2. $\int_{-\infty}^{\infty} \delta(x)\, dx = 1;$
3. for all functions $g(x)$ that are continuous at $x = 0$,

$$\int_{-\infty}^{\infty} g(x)\delta(x)\, dx = \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} g(x) D(x;\epsilon)\, dx = g(0) \,.$$

This last property is known as the *sampling property*. For any test function continuous at $x = 0$, the delta function samples its value there. The generalisation to functions $g(x)$ that are continuous at $x = x_0$ is (for $b > a$)

$$\int_a^b g(x)\delta(x - x_0)\, dx = \begin{cases} g(x_0) \text{ if } a < x_0 < b \\ 0 \text{ otherwise}\,. \end{cases}$$

5.2 Delta-function forcing

The delta function is a mathematically convenient way of expressing impulsive forcing terms.

Consider

$$y'' + p(x)y' + q(x)y = \delta(x)\,, \tag{33}$$

where $p(x)$ and $q(x)$ are continuous functions. For $x < 0$ and $x > 0$, $y(x)$ satisfies

$$y'' + p(x)y' + q(x)y = 0\,.$$

However, at $x = 0$ there will be a discontinuity in $y'(x)$.

We can see this by integrating Eq. (33) around a small interval $-\epsilon < x < \epsilon$ and taking the limit as $\epsilon \to 0$:

$$\lim_{\epsilon\to 0}\,[y']_{-\epsilon}^{\epsilon} + p(0)\lim_{\epsilon\to 0}\,[y]_{-\epsilon}^{\epsilon} + \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} qy\,dx = 1\,.$$

If we assume that y is continuous at $x = 0$, the second and third term on the left vanish leaving the *jump condition*

$$\lim_{\epsilon\to 0}\,[y']_{-\epsilon}^{\epsilon} = 1\,.$$

Our assumption that $y(x)$ has to be continuous at $x = 0$ can be established by contradiction. If it were discontinuous there, then the first derivative would be like a delta function and the second derivative even worse behaved making it impossible to satisfy the original differential equation.

Generally, discontinuities arising from delta-function forcing are addressed by the highest-order derivative appearing in the differential equation.

Example. Consider the *boundary-value* problem

$$y'' - y = 3\delta\left(x - \frac{\pi}{2}\right) \quad \text{with} \quad y = 0 \text{ at } x = 0 \text{ and } \pi\,.$$

We solve this by considering the regions $0 \le x < \pi/2$ and $\pi/2 < x \le \pi$ separately, and then join these together using the appropriate jump condition.

Away from $x = \pi/2$, we have

$$y'' - y = 0\,, \tag{34}$$

with general solution $y = A\sinh x + B\cosh x$. (It is more convenient to use hyperbolic functions rather than exponentials since we require $y(x)$ to vanish at the boundary points.) For $0 \le x < \pi/2$, the relevant solution is

$$y(x) = A\sinh x\,,$$

while for $\pi/2 < x \le \pi$ we have

$$y(x) = C\sinh(\pi - x)\,.$$

We now have to join these solutions up at $x = \pi/2$ such that y is continuous there and

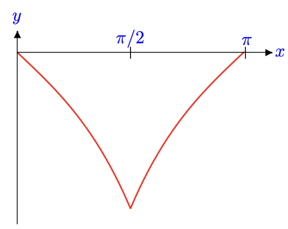
$$\lim_{\epsilon\to 0}\,[y']_{\pi/2-\epsilon}^{\pi/2+\epsilon} = 3\,.$$

Continuity of y implies that $A = C$, while the jump condition on y' gives

$$\begin{aligned} -C\cosh(\pi/2) - A\cosh(\pi/2) = 3 \\ \Rightarrow \quad A = C = \frac{-3}{2\cosh(\pi/2)}\,. \end{aligned}$$

The full solution is therefore

$$y(x) = \begin{cases} -\frac{3}{2}\frac{\sinh x}{\cosh(\pi/2)} & 0 \le x < \pi/2 \\ -\frac{3}{2}\frac{\sinh(\pi - x)}{\cosh(\pi/2)} & \pi/2 < x \le \pi\,. \end{cases}$$



This is shown in the figure to the right. Note, in particular, the discontinuity in the gradient at $x = \pi/2$.

5.3 Heaviside step function $H(x)$

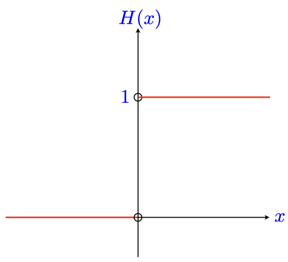
Definition (Heaviside step function). The *Heaviside step function* is defined as the integral of the delta function:

$$H(x) = \int_{-\infty}^{x} \delta(t)\,dt\,.$$

It follows that

$$H(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x > 0 \\ \text{undefined} & \text{at } x = 0\,. \end{cases}$$

The Heaviside function is shown schematically in the figure to the right. Note that it is undefined at $x = 0$. From the fundamental theorem of calculus, we have that



$$\frac{dH}{dx} = \delta(x)\,,$$

but, recall, that such functions and relationships can only be used inside integrals.

Generally, we see the smoothing effect of integration and the sharpening effect of differentiation: the derivative of the Heaviside function is very rapidly varying around the origin. On the other hand, the integral of the Heaviside function (sometimes called the *ramp function*) is continuous at $x = 0$ and just has a discontinuous first derivative.

5.3.1 Forcing with the Heaviside step function

The Heaviside step function can be used to describe situations where the forcing changes discontinuously. For example, if a switch is closed in an electrical circuit, the electromotive force of the battery is suddenly now applied to the other circuit components.

Consider a system described by

$$y'' + p(x)y' + q(x)y = H(x)\,,$$

where $p(x)$ and $q(x)$ are continuous at $x = 0$. For $x < 0$, $y(x)$ satisfies

$$y'' + p(x)y' + q(x)y = 0\,,$$

while for $x > 0$,

$$y'' + p(x)y' + q(x)y = 1\,.$$

The solutions of these equations are joined by noting that

$$\lim_{\epsilon \to 0} \left([y'']_{-\epsilon}^{\epsilon} + p(0)\, [y']_{-\epsilon}^{\epsilon} + q(0)\, [y]_{-\epsilon}^{\epsilon} \right) = 1\,.$$

If y'' goes like $H(x)$ in the vicinity of $x = 0$, then y' and y are both continuous there and the above condition is satisfied. We thus have the jump conditions:

$$\lim_{\epsilon \to 0} [y']_{-\epsilon}^{\epsilon} = 0 \quad \text{and} \quad \lim_{\epsilon \to 0} [y]_{-\epsilon}^{\epsilon} = 0\,.$$

6 Higher-order discrete (difference) equations

Much of what we have learnt about the solutions of linear differential equations goes over to discrete (difference) equations. Recall that we first met these in Topic II as approximations to first-order differential equations.

Consider a discrete second-order equation of the form

$$a y_{n+2} + b y_{n+1} + c y_n = f_n\,, \tag{35}$$

where a, b and c are constants. This now couples y_{n+2} to both y_{n+1} and y_n. Such couplings arise if we consider discretising a second-order differential equation at points $\{x_n\}$ spaced by h, since the second derivative at x_n can be approximated by

$$\left.\frac{d^2 y}{dx^2}\right|_{x_n} \approx \frac{y_{n+1} + y_{n-1} - 2y_n}{h^2}\,.$$

$$\lim_{h \to 0} \frac{y(x_n+h) + y(x_n-h) - 2y(x_n)}{h^2} = \left.\frac{d^2y}{dx^2}\right|_{x_n}$$

by Taylor series

We can solve Eq. (35) by exploiting linearity and eigenfunctions, as in the case of differential equations.

We first look for complementary functions that satisfy

$$a y_{n+2} + b y_{n+1} + c y_n = 0\,. \tag{36}$$

For a linear second-order differential equation with constant coefficients, the complementary functions take the form $y_c \propto e^{\lambda x}$. The discrete version of this is $y_n^{(c)} \propto k^n$

$x = nh$, $y = e^{\lambda n h} = k^n$

or $y_{n+1} = \lambda y_n \Rightarrow y_n = \lambda^n$.

f_n	$y_n^{(p)}$
k^n	Ak^n if $k \neq k_1$ or k_2
k_1^n	Ank_1^n
n^p (p a non-negative integer)	$An^p + Bn^{p-1} + \cdots + Cn + D$

Table 2: Form of particular "integrals" $y_n^{(p)}$ for discrete equations of the type in Eq. (35). Here, k_1 and k_2 are the roots of the characteristic equation for the homogeneous equation.

for some k to be determined. Trying this in Eq. (36), we have

$$\begin{aligned} ak^{n+2} + bk^{n+1} + ck^n = 0 \\ \Rightarrow \quad ak^2 + bk + c = 0\,. \end{aligned}$$

This characteristic equation has two roots in general, $k = k_1$ and $k = k_2$. The general complementary function is then

$$y_n^{(c)} = \begin{cases} Ak_1^n + Bk_2^n & \text{if } k_1 \neq k_2\,, \\ (A + Bn)k_1^n & \text{if } k_1 = k_2 = k \end{cases}.$$

The degenerate case, $k_1 = k_2 = k$, follows by analogy with degenerate differential equations: $xe^{\lambda x} \to nk^n$. B absorbs h term?

We can guess particular "integrals" of Eq. (35) for simple forcing sequences f_n; see Table 2.

Example (Fibonacci sequence). The Fibonacci sequence is defined by

$$y_n = y_{n-1} + y_{n-2}\,, \quad y_0 = y_1 = 1\,. \tag{37}$$

The sequence arises in all sorts of unexpected contexts. For example, in biological systems it arises in the arrangements of leaves on a stem or spikes on a pineapple. The first few elements in the sequence for $n = 0, 1, 2, 3, 4, 5$ are, of course, $y_n = 1, 1, 2, 3, 5, 8$.

We can rewrite Eq. (37) as

$$y_{n+2} - y_{n+1} - y_n = 0\,.$$

Trying $y_n = k^n$, we find

$$k^2 - k - 1 = 0 \quad \Rightarrow \quad k = \frac{1 \pm \sqrt{5}}{2}\,,$$

which you may recognise as the "golden ratio" or "golden mean" and (the negative of) its inverse:

$$\varphi_1 = \frac{1+\sqrt{5}}{2}\,, \qquad \varphi_2 = \frac{1-\sqrt{5}}{2} = \frac{-1}{\varphi_1}\,.$$

The solution of Eq. (37) is therefore of the form

$$y_n = A\varphi_1^n + B\varphi_2^n\,,$$

where A and B are given by the initial conditions

$$y_0 = 1 = A + B \quad \text{and} \quad y_1 = 1 = A\varphi_1 + B\varphi_2\,.$$

These are solved by

$$A = \frac{\varphi_1}{\sqrt{5}} \quad \text{and} \quad B = -\frac{\varphi_2}{\sqrt{5}}\,,$$

so that

$$y_n = \frac{\varphi_1^{n+1} - \varphi_2^{n+1}}{\sqrt{5}} = \frac{\varphi_1^{n+1} - (-1/\varphi_1)^{n+1}}{\sqrt{5}}\,.$$

This result is remarkable! It expresses a sequence of *integers* in terms of the difference of powers of the irrational golden ratio. Noting that $\varphi_1 > 1$, we have

$y_n \Rightarrow \frac{\varphi^n}{\sqrt{5}}$

$$\lim_{n\to\infty} y_{n+1}/y_n = \varphi_1\,,$$

so the ratio of adjacent terms of the Fibonacci sequence tends to the golden ratio.

7 Series solutions

In this section, we develop techniques to find series solutions to linear, homogeneous second-order differential equations when we (perhaps) cannot find simple closed forms. This builds on the brief discussion of series solutions for first-order differential equations in Topic II.

We consider equations of the form

$$p(x)y'' + q(x)y' + r(x)y = 0\,. \tag{38}$$

The feasibility of constructing a series solution in the vicinity of the point $x = x_0$ depends on the nature of the functions $p(x)$, $q(x)$ and $r(x)$ there.

7.1 Classification of singular points

Definition (Ordinary and singular points). The point $x = x_0$ is an *ordinary point* of the differential equation (38) if both $q(x)/p(x)$ and $r(x)/p(x)$ have Taylor series around $x = x_0$ (i.e., they are *analytic* there). Otherwise, $x = x_0$ is a *singular point.*

We want a Taylor series for y'', so we can calculate all derivatives given 0th and 1st.

We can classify singular points further as follows. If $x = x_0$ is a singular point, but Eq. (38) can be rewritten as

$$\overbrace{P(x)(x-x_0)^2}^{p(x)}y'' + \overbrace{Q(x)(x-x_0)}^{q(x)}y' + \overbrace{R(x)}^{r(x)}y = 0\,,$$

where $Q(x)/P(x)$ and $R(x)/P(x)$ do have Taylor series around $x = x_0$, then $x = x_0$ is a *regular singular point.* Otherwise, $x = x_0$ is an *irregular singular point.*

if regular singular point, it is no more singular than equidimensional equations, e.g. x^2 y'' + xy' + y = 0 has a regular singular point at x = 0.

Note that this condition for a regular singular point is equivalent to $(x - x_0)q(x)/p(x)$ and $(x - x_0)^2 r(x)/p(x)$ in Eq. (38) having Taylor series around $x = x_0$.

Lets us remove singular behaviour by removing a pole in p(x)

Loosely, for a regular singular point, the equation is singular, but not *too* singular, due to the properties of the derivative (cf. equidimensional equations).

Example. Consider

$$(1 - x^2)y'' - 2xy' + 2y = 0\,.$$

We have

$$\frac{q(x)}{p(x)} = \frac{-2x}{1-x^2} \quad \text{and} \quad \frac{r(x)}{p(x)} = \frac{2}{1-x^2}\,.$$

It follows that $x = \pm 1$ are singular points. However, as

$$(x-1)\frac{q(x)}{p(x)} = \frac{2x}{x+1} \quad \text{and} \quad (x-1)^2\frac{r(x)}{p(x)} = \frac{-2(x-1)}{x+1}\,,$$

for example, we see that these are both regular singular points.

Example. Consider

$$\sin x y'' + \cos x y' + 2y = 0\,.$$

Here,

$$\frac{q(x)}{p(x)} = \frac{\cos x}{\sin x} \quad \text{and} \quad \frac{r(x)}{p(x)} = \frac{2}{\sin x}\,.$$

Clearly, $x = n\pi$, for n an integer, are singular points while all others are ordinary points. Writing $x - n\pi = \epsilon$, we have

we're interested about x near n pi.

$$(x - n\pi)\frac{q(x)}{p(x)} = \epsilon\frac{\cos\epsilon}{\sin\epsilon}\,,$$

which is analytic at $\epsilon = 0$. Similarly, $(x - n\pi)^2 r(x)/p(x)$ is analytic at $x = n\pi$, so the points $x = n\pi$ are all regular singular points.

$= \dfrac{2\epsilon^2}{(-1)^n \sin\epsilon}$ ✓

For this equation, all other points are ordinary points.

Example. Consider

$$\left(1 + \sqrt{x}\right) y'' - 2xy' + 2y = 0\,.$$

We have

$$\frac{q(x)}{p(x)} = \frac{-2x}{1 + \sqrt{x}}\,,$$

which does not have a Taylor series around $x = 0$ (the second derivative is undefined). Similarly, $xq(x)/p(x)$ does not have a Taylor series there either so the point $x = 0$ is an irregular singular point.

7.2 Method of Frobenius

We now develop the series-expansion method of obtaining at least one solution of the linear, homogeneous, second-order differential equation (38). We can always do this provided the expansion point, $x = x_0$ is no worse than a regular singular point.

Theorem (Fuchs' theorem). If $x = x_0$ is an ordinary point of Eq. (38), then there are two linearly independent solutions of the form

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n\,,$$

i.e., in the form of a Taylor series, convergent in some neighbourhood of x_0.

If, instead, $x = x_0$ is an irregular singular point of Eq. (38), then there is at least one solution of the form

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+\sigma}, \qquad (39)$$

where σ is real and $a_0 \neq 0$ (so that σ is unique). This is an example of a *Frobenius series*. It can also be written as

$$y = (x - x_0)^{\sigma} \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

where the summation here is a Taylor series.

If we approximate our differential equation as an equidimensional equation about the singular point, we will get this solution where sigma is not necessarily an integer.

Note that when expanding about a regular singular point, there is no guarantee that one will obtain *two* linearly independent series solutions. We shall return to the construction of a second solution in such cases later.

Attempting a series solution about an irregular singular point may fail completely.

The method of Frobenius is best illustrated by example. Let us first consider an example of expanding about an ordinary point, and then a regular singular point.

Example (ordinary point). Let us consider again

$$(1 - x^2)y'' - 2xy' + 2y = 0, \qquad (40)$$

which we saw earlier has regular singular points at $x = \pm 1$ while all other points are ordinary. Expanding about $x = 0$ (an ordinary point), we try

$$y = \sum_{n=0}^{\infty} a_n x^n,$$

so that

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

It is convenient to multiply the original equation through by x^2 to make it like an equidimensional equation but with polynomial coefficients:

$$(1-x^2)\left(x^2y''\right) - 2x^2\left(xy'\right) + 2x^2y = 0\,.$$

Substituting for $y(x)$ and its derivatives, we have

$$\sum_{n=2}^{\infty} a_n\left[(1-x^2)n(n-1)\right]x^n - 2\sum_{n=1}^{\infty} a_n(x^2n)x^n + 2\sum_{n=0}^{\infty} a_n(x^2)x^n = 0\,.$$

Equating coefficients of x^n, for $n \geq 2$ we have

$$a_n n(n-1) - a_{n-2}(n-2)(n-3) - 2a_{n-2}(n-2) + 2a_{n-2} = 0\,,$$

or

$$n(n-1)a_n = n(n-3)a_{n-2} \qquad (n \geq 2)$$

$$\Rightarrow \quad a_n = \frac{n-3}{n-1}a_{n-2}\,. \tag{41}$$

This is a recurrence relation determining a_n in terms of a_{n-2}. Note that a_0 and a_1 are not fixed by this procedure – they are arbitrary constants set by the initial/boundary conditions for the differential equation.

From the recursion relation, we have $a_3 = 0$ and so $a_{2k+1} = 0$ for all $k \geq 1$. This gives one solution, $y = a_1x$. ($a_0 = 0$, $a_1 \neq 0$)

On the other hand, for n even, we have ($a_1 = 0$, $a_0 \neq 0$)

$$a_n = \frac{(n-3)}{(n-1)}a_{n-2} = \frac{(n-3)}{(n-1)}\frac{(n-5)}{(n-3)}a_{n-4} = \cdots$$
$$= -\frac{1}{n-1}a_0\,,$$

as terms alternately cancel. Therefore

$$y = a_0\left[1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} - \cdots\right]$$
$$= a_0\left[1 - \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right)\right]\,,$$

noting that

$$\ln(1 \pm x) = \pm x - \frac{x^2}{2} \pm \frac{x^3}{3} - \cdots .$$

Finally, the general solution of the differential equation (40) is

$$y(x) = a_1 x + a_0 \left[1 - \frac{x}{2} \ln\left(\frac{1+x}{1-x}\right)\right] .$$

Note the logarithmic behaviour of this solution at $x = \pm 1$, which, recall, are regular singular points of the differential equation. We shall return to this observation when discussing the construction of a second solution when expanding about a regular singular point.

Example (regular singular point). Consider

$$4xy'' + 2(1 - x^2)y' - xy = 0 .$$

For this equation, $x = 0$ is a regular singular point. Let us look for a series solution about this point. From Fuchs' theorem, we try

$$y = \sum_{n=0}^{\infty} a_n x^{n+\sigma} ,$$

with $a_0 \neq 0$. Again, it is convenient to multiply through the differential equation (this time by x) to write it as

$$4\left(x^2 y''\right) + 2(1 - x^2)\left(xy'\right) - x^2 y = 0 .$$

Substituting for y and its derivatives, we have

$$\sum_{n=0}^{\infty} a_n x^{n+\sigma} \left[4(n+\sigma)(n+\sigma-1) + 2(1-x^2)(n+\sigma) - x^2\right] = 0 .$$

As before, we equate coefficients of powers of x. The lowest power is x^σ with coefficient

$$a_0 \left[4\sigma(\sigma - 1) + 2\sigma\right] = 0$$
$$\Rightarrow \quad a_0 \sigma(2\sigma - 1) = 0 .$$

This is called the *indicial equation* for the *index* σ. Since $a_0 \neq 0$ by construction, we must have $\sigma = 0$ or $\sigma = 1/2$. Generally, the lowest power of x gives rise to the indicial equation, and its roots determine the index σ in the Frobenius series.

The next-lowest power is $x^{\sigma+1}$ with coefficient

$$\begin{aligned} & a_1 \left[4\sigma(\sigma+1) + 2(\sigma+1)\right] = 0 \\ \Rightarrow \quad & a_1(\sigma+1)(2\sigma+1) = 0\,. \end{aligned}$$

For both cases $\sigma = 0$ and $\sigma = 1/2$, this implies $a_1 = 0$.

Finally, we consider the power $x^{n+\sigma}$ for $n \geq 2$. This gives rise to the recursion relation

$$\begin{aligned} & \left[4(n+\sigma)(n+\sigma-1) + 2(n+\sigma)\right] a_n \\ & \qquad\qquad + \left[-2(n+\sigma-2) - 1\right] a_{n-2} = 0\,, \end{aligned}$$

which can be rearranged to obtain

$$2(n+\sigma)(2n+2\sigma-1)a_n = (2n+2\sigma-3)a_{n-2}\,. \quad (42)$$

We now consider the two roots of the indicial equation separately.

For $\sigma = 0$, the recursion relation (42) becomes

$$2n(2n-1)a_n = (2n-3)a_{n-2} \quad (n \geq 2)\,.$$

Since $a_1 = 0$, it follows that $a_{2k+1} = 0$ for all $k \geq 0$. For n even, we have

$$a_n = \frac{2n-3}{2n(2n-1)} a_{n-2}\,,$$

and so

$$a_2 = \frac{1}{4\times 3} a_0\,, \quad a_4 = \frac{5}{8\times 7} a_2 = \frac{5}{8\times 7} \times \frac{1}{4\times 3} a_0\,, \quad \text{etc.}$$

It follows that we have one solution

$$y = a_0 \left(1 + \frac{1}{4\times 3}x^2 + \frac{5\times 1}{8\times 7\times 4\times 3}x^4 + \cdots\right)\,.$$

Note that this is a Taylor series.

We now consider the root $\sigma = 1/2$. In this case, the recursion relation (42) becomes

$$\begin{aligned} 2n(2n+1)a_n &= (2n-2)a_{n-2} \quad (n \geq 2) \\ \Rightarrow \quad a_n &= \frac{n-1}{n(2n+1)} a_{n-2} \, . \end{aligned}$$

This gives

$$a_2 = \frac{1}{2 \times 5} a_0 \, , \quad a_4 = \frac{3}{4 \times 9} a_2 = \frac{3}{4 \times 9} \times \frac{1}{2 \times 5} a_0 \, , \quad \text{etc.}$$

It follows that we have a second solution

$$y = b_0 x^{1/2} \left(1 + \frac{1}{2 \times 5} x^2 + \frac{3 \times 1}{4 \times 9 \times 2 \times 5} x^4 + \cdots \right) ,$$

where we have relabelled a_0 to b_0 to distinguish it from the constant in the other solution.

We see that for this example we have generated two linearly independent solutions with the series-expansion method. However, this is not generally the case, as we now discuss.

7.3 Second solutions

When expanding around a regular singular point $x = x_0$, we are guaranteed to be able to construct one solution as a Frobenius series. Whether we can generate a second depends critically on the roots, σ_1 and σ_2, of the indicial equation. There are three cases to consider.

1. If $\sigma_2 - \sigma_1$ is not an integer, then there are always two linearly independent solutions:

   $$y = (x - x_0)^{\sigma_1} \sum_{n=0}^{\infty} a_n (x - x_0)^n \, ,$$

   $$y = (x - x_0)^{\sigma_2} \sum_{n=0}^{\infty} b_n (x - x_0)^n \, .$$

   Note that as $x \to x_0$, a linear combination of these goes like $a_0 (x - x_0)^{\sigma_1}$ if $\sigma_1 < \sigma_2$.

2. If $\sigma_2 - \sigma_1$ is a non-zero integer, there is one solution of the form

$$y_1 = (x - x_0)^{\sigma_2} \sum_{n=0}^{\infty} a_n (x - x_0)^n ,$$

involving the larger root σ_2 of the indicial equation. The smaller root σ_1 will generally not generate a valid series solution (though in some special cases it may). Instead, the second solution is of the form

$$y_2 = (x - x_0)^{\sigma_1} \sum_{n=0}^{\infty} b_n (x - x_0)^n + c y_1 \ln(x - x_0) .$$

Here c is usually non-zero, i.e., the solution must generally include a part involving the solution y_1 multiplied by $\ln(x - x_0)$. The constant c is not arbitrary – it is fixed in terms of the constant a_0 in the first series and b_0, leaving the general solution with two arbitrary constants, as required.

3. If $\sigma_1 = \sigma_2 = \sigma$, then the log term is *always* required, i.e., $c \neq 0$. The solutions are then

$$y_1 = (x - x_0)^{\sigma} \sum_{n=0}^{\infty} a_n (x - x_0)^n ,$$
$$y_2 = (x - x_0)^{\sigma} \sum_{n=0}^{\infty} b_n (x - x_0)^n + c y_1 \ln(x - x_0) .$$

Example (Case 2). Consider the differential equation

$$x^2 y'' - x y = 0 .$$

The point $x = 0$ is a regular singular point. As usual, we look for a solution

$$y = \sum_{n=0}^{\infty} a_n x^{n+\sigma} ,$$

with $a_0 \neq 0$. The differential equation then requires

$$\sum_{n=0}^{\infty} \left[a_n (n+\sigma)(n+\sigma-1) x^{n+\sigma} - a_n x^{n+\sigma+1} \right] = 0 .$$

The coefficient of the lowest power of x (x^σ) determines the indicial equation:

$$a_0\sigma(\sigma-1)=0 \quad \Rightarrow \quad \sigma=0 \quad \text{or} \quad \sigma=1\,,$$

since $a_0 \neq 0$. The coefficients of the higher powers of x give

$$a_n(n+\sigma)(n+\sigma-1)=a_{n-1} \quad (n\geq 1)\,. \qquad (43)$$

We see that we have two roots of the indicial equation that differ by an integer (i.e., Case 2 above).

First consider the larger root, $\sigma = 1$. The recursion relation (43) gives (for $n \geq 1$)

$$a_n=\frac{a_{n-1}}{n(n+1)} \quad \Rightarrow \quad a_n=\frac{a_0}{n!(n+1)!}\,,$$

giving a solution

$$y_1=a_0x\left(1+\frac{x}{2}+\frac{x^2}{12}+\frac{x^3}{144}+\cdots\right)\,.$$

Now consider the root $\sigma = 0$. The recursion relation gives

$$a_nn(n-1)=a_{n-1} \quad (n\geq 1)\,.$$

For $n = 1$, the left-hand side vanishes implying that $a_0 = 0$. However, this is a contradiction since we have required that $a_0 \neq 0$.

In this example, we see that the smaller root of the indicial equation does not generate a series solution to the differential equation. Instead, the solution will take the form

$$y_2=cy_1\ln x+\sum_{n=0}^{\infty}b_nx^n\,. \qquad (44)$$

Construction of the second solution (non-examinable)

There are several ways to construct the second solution. The most direct is simply to assume the trial solution (44) and substitute it into the differential equation.

Proceeding this way, we have

$$xy_2 = cy_1 x\ln x + \sum_{n=0}^{\infty} b_n x^{n+1}\,,$$

$$x^2 y_2'' = c\left(y_1'' x^2 \ln x + 2xy_1' - y_1\right) + \sum_{n=0}^{\infty} b_n n(n-1)x^n\,,$$

Recalling that y_1 is a solution of the differential equation, we require

$$2cxy_1' - cy_1 + \sum_{n=0}^{\infty} b_n\left[n(n-1)x^n - x^{n+1}\right] = 0\,. \tag{45}$$

Note how the log terms have cancelled.

Recalling that

$$y_1 = \sum_{n=0}^{\infty} a_n x^{n+1} \quad \text{with} \quad a_n = \frac{a_0}{n!(n+1)!}\,,$$

so that

$$xy_1' = \sum_{n=0}^{\infty} a_n (n+1)x^{n+1}\,,$$

Eq. (45) reduces to

$$c\sum_{n=0}^{\infty} a_n(2n+1)x^{n+1} + \sum_{n=0}^{\infty} b_n\left[n(n-1)x^n - x^{n+1}\right] = 0\,.$$

The coefficient of the lowest power of x (x^0) vanishes identically, while for $n \geq 1$ we must have

$$ca_{n-1}(2n-1) + n(n-1)b_n - b_{n-1} = 0\,. \tag{46}$$

For $n = 1$, Eq. (46) gives

$$ca_0 = b_0\,,$$

which determines the constant c in terms of a_0 and b_0, as noted earlier. Substituting $c = b_0/a_0$ and using the explicit form of the a_n, Eq. (46) then reduces to

$$\frac{(2n-1)}{n!(n-1)!}b_0 - b_{n-1} + n(n-1)b_n = 0 \quad (n \geq 1)\,.$$

In this recursion relation, we can choose b_0 and b_1 arbitrarily and then all b_n for $n > 1$ are determined linearly from b_0 and b_1.

By assumption, $b_0 \neq 0$. If we were to take $b_0 = 0$, then $c = 0$ and the log term vanishes in the trial solution (44). Moreover, the recursion relation would then give

$$b_n = \frac{b_1}{n!(n-1)!}\,,$$

and the trial solution would reduce to

$$y_2 = b_1 \sum_{n=1}^{\infty} \frac{x^n}{n!(n-1)!} = b_1 \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!(n+1)!}\,.$$

This is proportional to y_1 and so we have not generated a linearly independent second solution.

We therefore need only consider the case $b_1 = 0$, so that

$$b_2 = -\frac{3}{4}b_0\,, \quad b_3 = -\frac{7}{36}b_0\,, \quad \text{etc.}$$

The second solution is then

$$y_2 = b_0 \ln x \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!(n+1)!} + b_0\left(1 - \frac{3}{4}x^2 - \frac{7}{36}x^3 + \cdots\right)\,, \qquad (47)$$

where the first summation on the right is y_1/a_0 and we have used $c = b_0/a_0$.

Reduction of order

An alternative way to construct a second solution is to use the method of reduction of order (Sec. 2.1). This has the benefit of showing why the second solution has to have the form of the trial solution (44).

Recall that with this method, given a solution y_1, we look for a second solution in the form $y_2 = v(x)y_1$. Given the trial second solution (44), we expect that $v(x)$ will involve a log term. Substituting $y_2 = v(x)y_1$ into the differential equation

$$x^2 y_2'' - xy_2 = 0\,,$$

and using the fact that y_1 is also a solution, we must have

$$v''y_1 + 2v'y_1' = 0\,.$$

This means that $u \equiv v'$ satisfies

$$u'y_1 + 2uy_1' = 0 \quad \Rightarrow \quad \frac{u'}{u} = -2\frac{y_1'}{y_1}\,.$$

The solution of this is

$$\ln u = -2\ln y_1 + \ln B \quad \Rightarrow \quad u = v' = \frac{B}{y_1^2}\,,$$

where B is an arbitrary constant.

We now use the series solution for y_1, which we repeat here for convenience:

$$y_1 = a_0 x\left(1 + \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{144} + \cdots\right)\,.$$

It follows that

$$\begin{aligned} v' &= \frac{B}{a_0^2 x^2}\left(1 + \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{144} + \cdots\right)^{-2} \\ &= \frac{B}{a_0^2 x^2}\left(1 - x + \frac{7}{12}x^2 - \frac{19}{72}x^3 + \cdots\right)\,, \end{aligned}$$

where we have performed a binomial expansion. Let us write this as

$$v' = \frac{B}{a_0^2}\left(\frac{1}{x^2} - \frac{1}{x} + \sum_{n=0}^{\infty} B_n x^n\right),$$

where $B_0 = 7/12$, $B_1 = -19/72$, etc. Integrating, we find

$$v = \frac{B}{a_0^2}\left(-\frac{1}{x} - \ln x + \sum_{n=0}^{\infty} \frac{B_n}{n+1} x^{n+1}\right),$$

where we have ignored any constant of integration since it would just add a multiple of the first solution, y_1, to y_2.

The final step is to multiply by y_1, where, recall,

$$y_1 = a_0 x \sum_{n=0}^{\infty} \frac{x^n}{n!(n+1)!}.$$

This gives

$$\begin{aligned} y_2 &= -\frac{B}{a_0}\ln x \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!(n+1)!} + \frac{B}{a_0}\left(1 + \frac{x}{2} + \frac{x^2}{12} + \cdots\right) \\ &\qquad \times \left(-1 + \sum_{n=0}^{\infty} \frac{B_n}{n+1} x^{n+2}\right) \\ &= -\frac{B}{a_0}\ln x \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!(n+1)!} - \frac{B}{a_0}\left[1 + \frac{x}{2} + \left(\frac{1}{12} - B_0\right)x^2 + \cdots\right], \end{aligned}$$

which is exactly of the form of the trial second solution (44).

We can make contact with the second solution (47), derived above by direct substitution, by identifying $-B/a_0$ with b_0 and using $B_0 = 7/12$:

$$y_2 = b_0 \ln x \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!(n+1)!} + b_0\left(1 + \frac{x}{2} - \frac{x^2}{2} + \cdots\right).$$

The second series on the right differs from that in Eq. (47). However, they only differ by a series proportional to y_1. If we subtract $b_0(x + x^2/2 + \cdots)/2$, we get

$$y_2 = b_0 \ln x \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!(n+1)!} + b_0\left(1 - \frac{3}{4}x^2 + \cdots\right),$$

which is exactly the second solution obtained previously.

V. MULTIVARIATE FUNCTIONS: APPLICATIONS

In this final topic, we shall consider functions of more than one variable. We shall introduce the idea of the *gradient vector*, which encodes the rate of change of the function along any direction, and see how to locate and classify the local extrema of multivariate functions.

We shall also look at systems of coupled, first-order differential equations where we have multiple dependent variables coupled together.

Finally, we shall briefly introduce the idea of *partial differential equations*, which are differential equations that describe the dynamics of multivariate functions.

1 Directional derivative

Consider a function $f(x, y)$, and an infinitesimal (vector) displacement $d\mathbf{s} = (dx, dy)$. The change in $f(x, y)$ due to the displacement is given straightforwardly by the multivariate chain rule:

$$\begin{aligned} df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \\ &= (dx, dy) \cdot \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \\ &= d\mathbf{s} \cdot \boldsymbol{\nabla} f\,. \end{aligned}$$

Here, the vector $\boldsymbol{\nabla} f$ is the *gradient* of f, also called grad$f$, with Cartestian components

$$\boldsymbol{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\,. \tag{1}$$

If we write the displacement as $d\mathbf{s} = ds\,\hat{\mathbf{s}}$, with $|\hat{\mathbf{s}}| = 1$, so that $\hat{\mathbf{s}}$ is the direction and ds is the distance moved, then

$$df = ds\,(\hat{\mathbf{s}} \cdot \boldsymbol{\nabla} f)\,.$$

This motivates introducing the *directional derivative* as follows.

Definition (Directional derivative). The *directional derivative* of f in the direction of $\hat{\mathbf{s}}$ is

$$\frac{df}{ds} = \hat{\mathbf{s}} \cdot \boldsymbol{\nabla} f \, .$$

It is the rate of change of f with distance along the direction $\hat{\mathbf{s}}$.

The directional derivative can be used to give an alternative, geometric definition of the gradient vector $\boldsymbol{\nabla} f$.

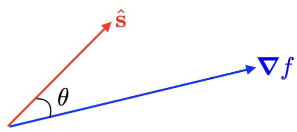
Definition (Gradient vector). The *gradient vector* $\boldsymbol{\nabla} f$ of the function f is defined as the vector that satisfies

$$\frac{df}{ds} = \hat{\mathbf{s}} \cdot \boldsymbol{\nabla} f$$

for all unit vectors $\hat{\mathbf{s}}$.

In Cartesian coordinates, where $d\mathbf{s} = (dx, dy)$, the components of the gradient vector reduce to Eq. (1).

If θ is the angle between $\hat{\mathbf{s}}$ and $\boldsymbol{\nabla} f$ (see figure to the right), we have



$$\frac{df}{ds} = \cos\theta \, |\boldsymbol{\nabla} f| \, .$$

We note from this result the following properties of the gradient vector.

1. The *direction* of $\boldsymbol{\nabla} f$ is the direction in which f *increases* most rapidly.
2. The *magnitude* of $\boldsymbol{\nabla} f$ is the maximum rate of change of f:
$$|\boldsymbol{\nabla} f| = \max_{\forall\theta} \left(\frac{df}{ds} \right) \, .$$
3. If $\hat{\mathbf{s}}$ is parallel to contours of f, then
$$0 = \frac{df}{ds} = \hat{\mathbf{s}} \cdot \boldsymbol{\nabla} f \, .$$
Hence, $\boldsymbol{\nabla} f$ is perpendicular to contours of $f(x, y)$.

2 Stationary points

There is always at least one direction in which $df/ds = 0$, i.e., tangent to the local contour of f.

Stationary points have $df/ds = 0$ for *all* directions. Since

$$\frac{df}{ds} = \hat{\mathbf{s}} \cdot \boldsymbol{\nabla} f \, ,$$

we must have

$$\boldsymbol{\nabla} f = 0 \quad \text{at stationary points.}$$

Stationary points may be local maxima, local minima or saddle points.

Near *local maxima*, the contours of f are locally elliptical (see the top row of Fig. 1). The gradient vector points towards a local maximum.

Near *local minima*, the contours of f are also locally elliptical (see the middle row of Fig. 1). The gradient vector points away from a local minimum.

Saddle points are stationary points that are neither local maxima nor minima. Near saddle points, the contours of f are locally hyperbolic (see the bottom row of Fig. 1). Also, the contour lines of f cross at and only at saddle points.

3 Classification of stationary points

To determine whether a stationary point is a maximum, minimum or saddle point, we consider the behaviour of the function in the vicinity of the point. To do so, it is useful first to consider how to generalise Taylor expansions to multivariate functions.

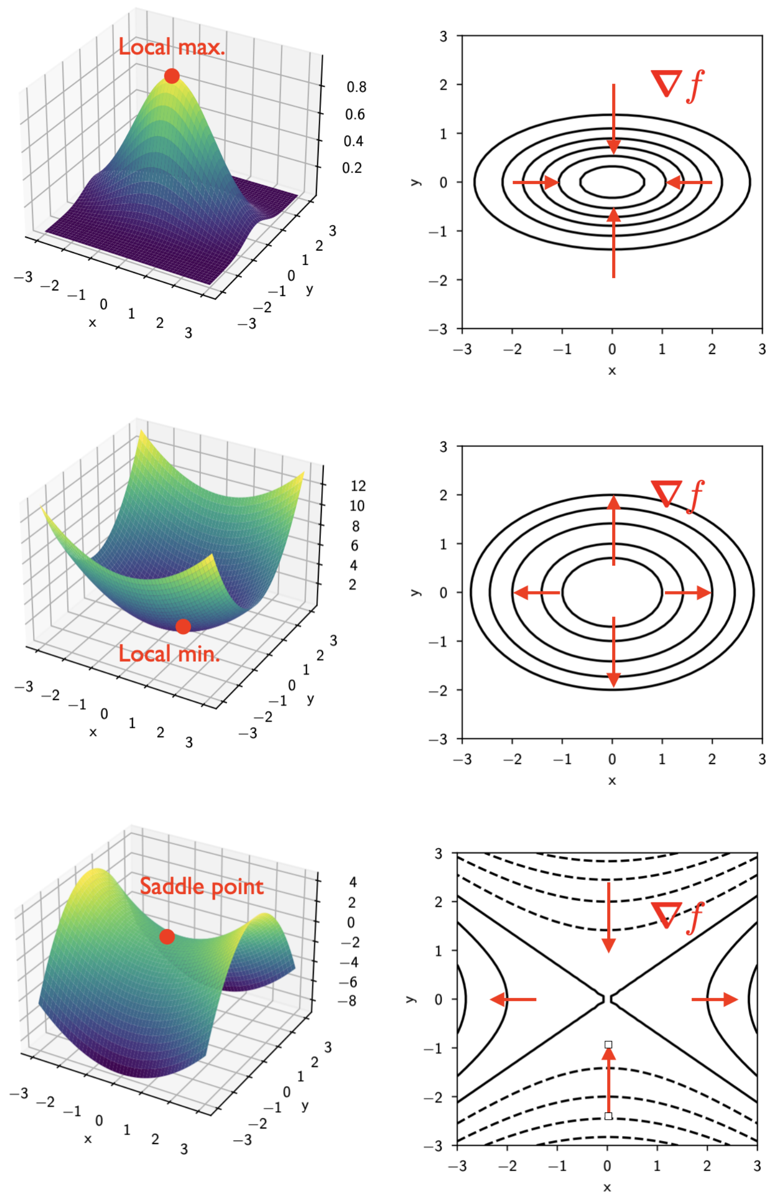


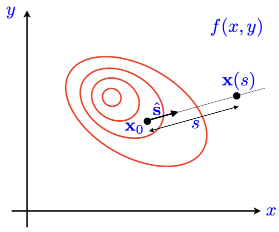
Figure 1: Illustrations of a local maximum (top), minimum (middle) and saddle point (bottom). The plots on the right show the contours near the stationary point and the gradient vector.

3.1 Taylor series for multivariate functions

Consider how a function $f(x, y)$ varies in the vicinity of the point $\mathbf{x}_0 = (x_0, y_0)$ as we move along the straight line through $\mathbf{x}_0$ in the direction of $\hat{\mathbf{s}}$. At distance s along this line from $\mathbf{x}_0$, we are at position

$$\mathbf{x}(s) = \mathbf{x}_0 + s\hat{\mathbf{s}}\,;$$

see the figure to the right.



Along this line, the function can be thought of as a function of s and the usual single-variable Taylor series holds, with the derivatives replaced by the directional derivative

$$\frac{df}{ds} = \hat{\mathbf{s}} \cdot \boldsymbol{\nabla} f\,.$$

It follows that

$$\begin{aligned} f(\mathbf{x}_0 + s\hat{\mathbf{s}}) &= f(\mathbf{x}_0) + s\left.\frac{df}{ds}\right|_{\mathbf{x}_0} + \frac{1}{2}s^2\left.\frac{d^2f}{ds^2}\right|_{\mathbf{x}_0} + \cdots \\ &= f(\mathbf{x}_0) + s\,\hat{\mathbf{s}}\cdot\boldsymbol{\nabla} f|_{\mathbf{x}_0} + \frac{1}{2}s^2\,(\hat{\mathbf{s}}\cdot\boldsymbol{\nabla})(\hat{\mathbf{s}}\cdot\boldsymbol{\nabla})f|_{\mathbf{x}_0} \\ &\qquad + \cdots\,. \end{aligned}$$

Let us write the finite displacement

$$\delta\mathbf{x} = s\hat{\mathbf{s}}\,,$$

with components $\delta x = x(s) - x_0$ and $\delta y = y(s) - y_0$. Then

$$s\hat{\mathbf{s}}\cdot\boldsymbol{\nabla} f = (\delta\mathbf{x})\cdot\boldsymbol{\nabla} f = (\delta x)\frac{\partial f}{\partial x} + (\delta y)\frac{\partial f}{\partial y}$$

and

$$\begin{aligned} s^2(\hat{\mathbf{s}}\cdot\boldsymbol{\nabla})(\hat{\mathbf{s}}\cdot\boldsymbol{\nabla})f &= (\delta\mathbf{x}\cdot\boldsymbol{\nabla})(\delta\mathbf{x}\cdot\boldsymbol{\nabla})f \\ &= \left(\delta x\frac{\partial}{\partial x} + \delta y\frac{\partial}{\partial y}\right)\left(\delta x\frac{\partial f}{\partial x} + \delta y\frac{\partial f}{\partial y}\right) \\ &= (\delta x)^2\frac{\partial^2 f}{\partial x^2} + 2\delta x\delta y\frac{\partial^2 f}{\partial x\partial y} + (\delta y)^2\frac{\partial^2 f}{\partial y^2} \\ &= (\delta x, \delta y)\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy}\end{pmatrix}\begin{pmatrix}\delta x \\ \delta y\end{pmatrix}. \end{aligned}$$

The matrix that appears here in the final line is the *Hessian matrix.*

Definition (Hessian matrix). The *Hessian matrix* is the matrix of second derivatives

$$\mathbf{H} \equiv \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \boldsymbol{\nabla}\boldsymbol{\nabla} f\,.$$

The Hessian is a symmetric matrix since partial derivatives commute, i.e., $f_{xy} = f_{yx}$.

Putting these results together, we have the multivariate Taylor series

$$f(x_0+\delta x, y_0+\delta y) = f(x_0, y_0) + \left(\delta x\frac{\partial f}{\partial x} + \delta y\frac{\partial f}{\partial y}\right)\Bigg|_{x_0,y_0} + \frac{1}{2}\left((\delta x)^2\frac{\partial^2 f}{\partial x^2} + 2\delta x\delta y\frac{\partial^2 f}{\partial x\partial y} + (\delta y)^2\frac{\partial^2 f}{\partial y^2}\right)\Bigg|_{x_0,y_0} + \cdots\,.$$

We can also write this in coordinate-free form as

$$f(\mathbf{x}_0+\delta\mathbf{x}) = f(\mathbf{x}_0)+\delta\mathbf{x}\cdot(\boldsymbol{\nabla} f)|_{\mathbf{x}_0}+\frac{1}{2}\delta\mathbf{x}\,(\boldsymbol{\nabla}\boldsymbol{\nabla} f)|_{\mathbf{x}_0}\,\delta\mathbf{x}^T + \cdots\,.$$

3.2 Nature of stationary points and the Hessian

Recall that for functions of one variable, e.g., $f(x)$, if the second derivative $d^2f/dx^2 > 0$ at a stationary point then it is a minimum, while if $d^2f/dx^2 < 0$ it is a maximum. In this section, we develop the equivalent results for multivariate functions.

At a stationary point $\mathbf{x}_0$, we know that $\boldsymbol{\nabla} f = 0$. It follows that in the vicinity of $\mathbf{x}_0$, we have

$$f(\mathbf{x}) = f(\mathbf{x}_0) + \frac{1}{2}\delta\mathbf{x}\,\mathbf{H}\,\delta\mathbf{x}^T + \cdots\,, \qquad (2)$$

where $\delta\mathbf{x} = \mathbf{x}-\mathbf{x}_0$ and the Hessian matrix $\mathbf{H}$ is evaluated at $\mathbf{x}_0$.

Definition (Positive-definite and negative-definite matrices). A (real) symmetric matrix H is *positive definite* if

$$\mathbf{x}\,\mathsf{H}\,\mathbf{x}^T > 0$$

for all non-zero (real) row vectors $\mathbf{x}$. Similarly, H is *negative definite* if

$$\mathbf{x}\,\mathsf{H}\,\mathbf{x}^T < 0$$

for all such $\mathbf{x}$. A matrix that is neither postive definite nor negative definite is sometimes called *indefinite*.

It follows that if the Hessian matrix is positive definite at a stationary point, then $\delta\mathbf{x}\,\mathsf{H}\,\delta\mathbf{x}^T > 0$ for all non-zero $\delta\mathbf{x}$. Equation (2) then implies that $f(\mathbf{x}) > f(\mathbf{x}_0)$ for all $\mathbf{x}$ sufficiently close to $\mathbf{x}_0$. This is just the definition of a local minimum, so we see that

$$\mathsf{H} \text{ positive definite} \Rightarrow \text{local minimum.}$$

Similarly, if H is negative definite, then $\delta\mathbf{x}\,\mathsf{H}\,\delta\mathbf{x}^T < 0$ for all non-zero $\delta\mathbf{x}$ and so $f(\mathbf{x}) < f(\mathbf{x}_0)$ in the vicinity of $\mathbf{x}_0$. The stationary point is therefore a local maximum:

$$\mathsf{H} \text{ negative definite} \Rightarrow \text{local maximum.}$$

If the matrix is indefinite, the stationary point may be a maximum, minimum or saddle (see below).

3.2.1 Definiteness and the eigenvalues

How can we determine whether a symmetric matrix is positive definite, negative definite or indefinite? As you know from *Vectors and Matrices*, any real symmetric matrix can be diagonalised by a suitable orthogonal transformation. Using coordinates along the principal axes, we have

$$\delta\mathbf{x}\,\mathsf{H}\,\delta\mathbf{x}^T = (\delta x_1, \delta x_2, \ldots, \delta x_n) \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \begin{pmatrix} \delta x_1 \\ \delta x_2 \\ \vdots \\ \delta x_n \end{pmatrix},$$

where we have generalised to a function of n variables. The eigenvalues $\{\lambda_i\}$ are real since the matrix H is real symmetric.

If $\delta\mathbf{x}\,\mathsf{H}\,\delta\mathbf{x}^T > 0$ for all non-zero $\delta\mathbf{x}$, we see that we need all the eigenvalues to be positive. It follows that

$$\mathsf{H} \text{ positive definite} \Leftrightarrow \text{all } \lambda_i > 0.$$

Similarly,

$$\mathsf{H} \text{ negative definite} \Leftrightarrow \text{all } \lambda_i < 0.$$

On the other hand, if all the eigenvalues are non-zero but have mixed signs, then $\delta\mathbf{x}\,\mathsf{H}\,\delta\mathbf{x}^T$ can be positive, negative or zero depending on the direction. This case corresponds to the stationary point being a saddle point.

If any of the eigenvalues of the Hessian are zero, further analysis (e.g., higher terms in the Taylor series) is needed to determine the nature of the stationary point. For example, the function

$$f(x, y) = x^2 + y^4$$

has a (global) minimum at $x = 0$, $y = 0$. The Hessian is

$$\mathsf{H}(x, y) = \begin{pmatrix} 2 & 0 \\ 0 & 12y^2 \end{pmatrix}$$

and so reduces to diag$(2, 0)$ at the stationary point. The eigenvalues there are 2 and 0.

3.2.2 Definiteness and the signature of the Hessian

There is an alternative way to establish if the Hessian matrix is positive or negative definite, using what is called the *signature* of the matrix. This avoids having to calculate the eigenvalues directly.

Definition (Signature of Hessian matrix). The *signature* of $\mathbf{H}$ is the pattern of the signs of the ordered subdeterminants of its leading principal minors. For a function of n variables, $f(x_1, x_2, \ldots, x_n)$, these subdeterminants are

$$\underbrace{f_{x_1x_1}}_{|\mathbf{H}_1|}, \underbrace{\begin{vmatrix} f_{x_1x_1} & f_{x_1x_2} \\ f_{x_2x_1} & f_{x_2x_2} \end{vmatrix}}_{|\mathbf{H}_2|}, \underbrace{\begin{vmatrix} f_{x_1x_1} & f_{x_1x_2} & f_{x_1x_3} \\ f_{x_2x_1} & f_{x_2x_2} & f_{x_2x_3} \\ f_{x_3x_1} & f_{x_3x_2} & f_{x_3x_3} \end{vmatrix}}_{|\mathbf{H}_3|}, \ldots, |\mathbf{H}_n| = |\mathbf{H}|.$$

It can be shown (*Sylvester's criterion*) that

$$\mathbf{H} \text{ positive definite} \Leftrightarrow \text{signature is } +, +, \ldots, +$$

and

$$\mathbf{H} \text{ negative definite} \Leftrightarrow \text{signature is } -, +, \ldots, (-1)^n.$$

It is straightforward to establish the forward implications here, for example that $\mathbf{H}$ being positive definite implies the $+, +, \ldots, +$ signature. If $\mathbf{H}$ is positive definite, then so too are all its principal minors. This follows from considering the quadratic form $\mathbf{x}\,\mathbf{H}\,\mathbf{x}^T > 0$ for vectors of the form $\mathbf{x} = (x_1, x_2, 0, \ldots, 0)$, for example. In this case, only the leading principal minor $\mathbf{H}_2$ is involved and so it must also be positive definite.[1] As all the principal minors are positive definite, they all have only positive eigenvalues and hence each has positive determinant. This establishes that the signature is $+, +, \ldots, +$. Similarly, if $\mathbf{H}$ is negative definite, so too are all its leading principal minors. It follows that all have only negative eigenvalues, and so the sign of $|\mathbf{H}_m|$ for $m = 1, \ldots, n$ is $(-1)^m$.

It takes more work to prove the converses in Sylvester's criterion (see non-examinable section below).

[1] This result means that if the quadratic function $\mathbf{x}\,\mathbf{H}\,\mathbf{x}^T$ has a minimum at $\mathbf{x} = 0$, it is also a minimum when the function is restricted to any lower-dimensional subspace that includes the origin. (In two dimensions, these would be straight lines through the origin.)

Sylvester's criterion (non-examinable)

Let us first sketch the proof of the converse in Sylvester's criterion for the positive-definite case. We aim to show that if the subdeterminants of the leading principal minors of a real-symmetric $n \times n$ matrix H are all positive, then H is positive definite. We start with H_1, which has a single element h_{11}. If $|\mathsf{H}_1| > 0$, then $h_{11} > 0$ and H_1 is positive definite.

We next show that if H_k is positive definite, and $|\mathsf{H}_{k+1}| > 0$, then H_{k+1} is also positive definite. If $|\mathsf{H}_{k+1}| > 0$, then its eigenvalues are either all positive, or all but two, four, etc., are positive and two, four, etc., are negative. We shall prove that it is not possible to have two or more negative eigenvalues by contradiction. Suppose that H_{k+1} does have two or more negative eigenvalues. Let two of the associated eigenvectors be $\mathbf{u}$ and $\mathbf{v}$, with components u_i and v_i for $i = 1, 2, \ldots, k+1$. Since these are the eigenvectors of a real-symmetric matrix, they may always be chosen to be orthogonal. Consider now the (row) vector

$$\mathbf{w} = v_{k+1}\mathbf{u} - u_{k+1}\mathbf{v}\,,$$

which by construction has no $k+1$ component. It follows that

$$\mathbf{w}\,\mathsf{H}_{k+1}\,\mathbf{w}^T = (v_{k+1})^2\,\mathbf{u}\,\mathsf{H}_{k+1}\,\mathbf{u}^T + (u_{k+1})^2\,\mathbf{v}\,\mathsf{H}_{k+1}\,\mathbf{v}^T < 0\,,$$

since $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors of H_{k+1} with negative eigenvalues. However, since $\mathbf{w}$ has no $k+1$ component, evaluating $\mathbf{w}\,\mathsf{H}_{k+1}\,\mathbf{w}^T$ amounts to using $(w_1, w_2, \cdots, w_k)$ in the quadratic form constructed from H_k. As H_k is positive definite, $\mathbf{w}\,\mathsf{H}_{k+1}\,\mathbf{w}^T > 0$ so we have a contradiction. It follows that all the eigenvalues of H_{k+1} are positive and so it is a positive-definite matrix.

Working through H_1, H_2 up to $\mathsf{H}_n = \mathsf{H}$, we see that if all have positive determinant, then they are all positive definite too. This establishes the converse in Sylvester's criterion for the positive-definite case.

To prove the negative-definite case, suppose that the determinants of the leading principal minors of the real-symmetric $n \times n$ matrix H have signature $-, +, \ldots, (-1)^n$. Consider the matrix $-\mathsf{H}$. Since multiplying a $k \times k$ matrix by -1 changes the determinant by $(-1)^k$, the matrix $-\mathsf{H}$ will have signature $+, +, \ldots, +$. As we have shown above, such a matrix must be positive definite. As $-\mathsf{H}$ is positive definite, H must be negative definite.

3.3 Contours near stationary points

Consider a function $f(x, y)$ with a stationary point at (x_0, y_0). Denote the Hessian matrix there by H, and adopt coordinates with axes aligned with the principal axes of $\mathbf{H}$. In these coordinates,

$$\mathbf{H} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

where λ_1 and λ_2 are the eigenvalues, which we shall assume are non-zero. If we write

$$\mathbf{x} = \mathbf{x}_0 + (\xi, \eta),$$

then in the vicinity of $\mathbf{x}_0$

$$f(\mathbf{x}) \approx f(\mathbf{x}_0) + \frac{1}{2}\left(\lambda_1 \xi^2 + \lambda_2 \eta^2\right).$$

The contours of f therefore locally satisfy

$$\lambda_1 \xi^2 + \lambda_2 \eta^2 = \text{const.} \tag{3}$$

At a maximum or minimum, the eigenvaluesall have the same sign so the contours given by Eq. (3) are locally *elliptical*. At a saddle point, the eigenvalues have opposite signs and so the contours are locally *hyperbolic*.

Example. Consider the function

$$f(x, y) = 4x^3 - 12xy + y^2 + 10y + 6.$$

We have

$$f_x = 12x^2 - 12y,$$
$$f_y = -12x + 2y + 10.$$

At stationary points, $f_x = 0$ and $f_y = 0$. The first of these gives $y = x^2$ and the second $6x = y + 5$. Substituting for $y = x^2$ we have

$$x^2 - 6x + 5 = 0 \quad \Rightarrow \quad x = 1 \text{ or } x = 5.$$

The stationary points are therefore $(1, 1)$ and $(5, 25)$.

Evaluating the second derivatives, we have

$$f_{xx} = 24x,$$
$$f_{xy} = -12,$$
$$f_{yy} = 2.$$

Consider first the point $(1,1)$. The Hessian there is

$$\mathbf{H} = \begin{pmatrix} 24 & -12 \\ -12 & 2 \end{pmatrix} .$$

The subdeterminants of the leading principal minors are $|\mathbf{H}_1| = 24$ and $|\mathbf{H}_2| = |\mathbf{H}| = -96$. The signature is $+,-$ and so $\mathbf{H}$ is neither positive definite nor negative definite. In this two-dimensional case, we know from $|\mathbf{H}| < 0$ that the eigenvalues are opposite in sign and so we have a saddle point.

At the other stationary point, $(5, 25)$, we have

$$\mathbf{H} = \begin{pmatrix} 120 & -12 \\ -12 & 2 \end{pmatrix} .$$

The subdeterminants of the leading principal minors are now $|\mathbf{H}_1| = 120$ and $|\mathbf{H}_2| = |\mathbf{H}| = 96$. The signature is $+,+$ and so we know from Sylvester's crterion that $\mathbf{H}$ is positive definite. We see that $(5, 25)$ is a local minimum.

To determine the orientation of the contours near the stationary points, consider, for example, the saddle point $(1,1)$. Writing

$$(x, y) = (1, 1) + (\delta x, \delta y) ,$$

the contours locally have

$$\begin{aligned} & f_{xx}(\delta x)^2 + 2 f_{xy}\delta x \delta y + f_{yy}(\delta y)^2 = \text{const.} \\ \Rightarrow \quad & 12(\delta x)^2 - 12\delta x \delta y + (\delta y)^2 = \text{const.} \end{aligned}$$

The intersecting straight-line contours through the saddle point (which are also the asymptotes of the neighbouring hyperbolic contours) are therefore described by

$$12(\delta x)^2 - 12\delta x\delta y + (\delta y)^2 = 0 \quad \Rightarrow \quad \delta y = (6 \pm 2\sqrt{6})\delta x .$$

To sketch the contours, we can draw what they look like near the stationary points and then try to join them togther noting they only cross at the saddle point. The contours are shown in Fig. 2.

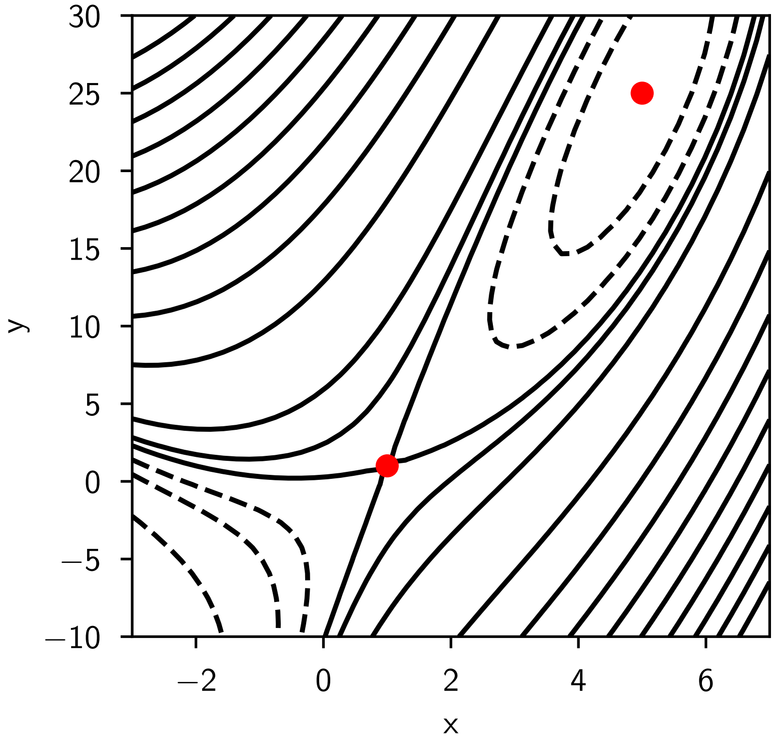


Figure 2: Contours of the function $f(x,y) = 4x^3 - 12xy + y^2 + 10y + 6$. (The contour levels are not equally spaced.) Note the shape of the contours close to the saddle point at $(1,1)$ and the local minimum at $(5,25)$.

4 Systems of linear differential equations

In this section we consider the behaviour of systems of first-order linear differential equations, where we have multiple dependent variables that may be coupled to each other.

Consider two functions, $y_1(t)$ and $y_2(t)$, which satisfy

$$\dot{y}_1 = ay_1 + by_2 + f_1(t)\,, \tag{4}$$
$$\dot{y}_2 = cy_1 + dy_2 + f_2(t)\,, \tag{5}$$

where a, b, c and d are constants. We can write these in vector form as

$$\dot{\mathbf{Y}} = \mathsf{M}\mathbf{Y} + \mathbf{F}\,,$$

where

$$\mathbf{Y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad \mathsf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

One way to solve Eqs (4) and (5) is to convert them into a higher-order equation for one of the dependent variables. For example, differentiating Eq. (4), we have

$$\begin{aligned} \ddot{y}_1 &= a\dot{y}_1 + b\dot{y}_2 + \dot{f}_1 \\ &= a\dot{y}_1 + b\,(cy_1 + dy_2 + f_2) + \dot{f}_1 \\ &= a\dot{y}_1 + bcy_1 + d\,(\dot{y}_1 - ay_1 - f_1) + bf_2 + \dot{f}_1\,, \end{aligned}$$

so that

$$\ddot{y}_1 - (a+d)\dot{y}_1 + (ad - bc)y_1 = bf_2 - df_1 + \dot{f}_1\,.$$

This is a linear, second-order differential equation with constant coefficients, which we know how to solve.

However, it is often more convenient to solve the first-order system of equations directly with the matrix methods developed in the rest of this section, rather than solving the higher-order equation. Indeed, we often go the other way and convert a linear higher-order differential equation to a system of (coupled) linear first-order equations. This is particularly the case when solving equations numerically.

For example, the second-order equation

$$\ddot{y} + \alpha\dot{y} + \beta y = f\,,$$

where α and β are constant coefficients, can be recast as a first-order system by writing

$$y_1 = y \quad \text{and} \quad y_2 = \dot{y}\,,$$

so that

$$\begin{aligned} \dot{y}_1 &= y_2\,, \\ \dot{y}_2 &= \ddot{y} = -\alpha y_2 - \beta y_1 + f\,. \end{aligned}$$

In matrix form, this is

$$\dot{\mathbf{Y}} = \begin{pmatrix} 0 & 1 \\ -\beta & -\alpha \end{pmatrix} \mathbf{Y} + \begin{pmatrix} 0 \\ f \end{pmatrix}$$

with $\mathbf{Y} = (y_1, y_2)^T$.

4.1 Matrix methods

To solve a linear system of equations of the form

$$\dot{\mathbf{Y}} = \mathsf{M}\mathbf{Y} + \mathbf{F}(t)\,,$$

where the matrix M has constant elements, we proceed as follows.

1. We write $\mathbf{Y} = \mathbf{Y}_c + \mathbf{Y}_p$, where the complementary solution $\mathbf{Y}_c$ satisfies the homogeneous equation

$$\dot{\mathbf{Y}}_c = \mathsf{M}\mathbf{Y}_c\,. \tag{6}$$

2. We look for a complementary solution of the form $\mathbf{Y}_c = \mathbf{v}e^{\lambda t}$, where $\mathbf{v}$ is a constant vector. For this to satisfy Eq. (6), we must have

$$\mathsf{M}\mathbf{v} = \lambda\mathbf{v}\,,$$

i.e., $\mathbf{v}$ must be an eigenvector of M and then λ is the associated eigenvalue. For a system of n equations, there will be n such complementary solutions if the eigenvalues are all distinct (in which case there are n linearly independent eigenvectors). Any linear combination of these is a solution of Eq. (6).

We will only consider the non-degenerate cases where we have n distinct eigenvalues.

3. Finally, we find a particular solution, $\mathbf{Y}_p$, which satisfies the full system of forced equations. Its form will depend on the forcing vector $\mathbf{F}(t)$.

Example. Consider the linear system

$$\dot{\mathbf{Y}} = \underbrace{\begin{pmatrix} -4 & 24 \\ 1 & -2 \end{pmatrix}}_{\mathsf{M}} \mathbf{Y} + \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^t \,. \tag{7}$$

We look for a complementary solution of the form $\mathbf{Y}_c = \mathbf{v}e^{\lambda t}$. The eigenvalues λ of the matrix M follow from $\det(\mathsf{M} - \lambda \mathsf{I}) = 0$, which gives

$$(\lambda + 8)(\lambda - 2) = 0 \quad \Rightarrow \quad \lambda = 2, -8\,.$$

The associated eigenvectors are

$$\mathbf{v}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad \text{for } \lambda_1 = 2$$

and

$$\mathbf{v}_2 = \begin{pmatrix} -6 \\ 1 \end{pmatrix} \quad \text{for } \lambda_2 = -8$$

The general complementary solution is therefore

$$\mathbf{Y}_c = A \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{2t} + B \begin{pmatrix} -6 \\ 1 \end{pmatrix} e^{-8t} \,,$$

where A and B are constants.

For the particular solution, we try $\mathbf{Y}_p = \mathbf{u}e^t$, inspired by the time dependence of the forcing term. We require

$$\mathbf{u} = \mathsf{M}\mathbf{u} + \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} 5 & -24 \\ -1 & 3 \end{pmatrix} \mathbf{u} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \rightarrow \quad u = -(M - I)^{-1} \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad \mathbf{u} = -\frac{1}{9} \begin{pmatrix} 3 & 24 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = -\begin{pmatrix} 4 \\ 1 \end{pmatrix} .$$

1 is not an eigenvalue, so the inverse exists

It follows that the general solution of the full system is

$$\mathbf{Y} = A\begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{2t} + B\begin{pmatrix} -6 \\ 1 \end{pmatrix} e^{-8t} - \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{t}.$$

Note that if the time dependence of the forcing is $e^{\lambda t}$, where λ is an eigenvalue of M, then we should instead look for a particular solution

$$\mathbf{Y}_p = \mathbf{u}te^{\lambda t}.$$

4.2 Non-degenerate phase portraits

The phase space for a system of n first-order differential equations is the n-dimensional space with points $\mathbf{Y} = (y_1, y_2, \ldots, y_n)^T$.

A phase portrait shows the solution trajectories in this space.

We shall consider the homogeneous equation

$$\dot{\mathbf{Y}} = \mathsf{M}\mathbf{Y},$$

which clearly has a fixed point at $\mathbf{Y} = 0$. For $n = 2$, the general solution of the equation in the *non-degenerate* case, $\lambda_1 \neq \lambda_2$, is

$$\mathbf{Y}(t) = \mathbf{v}_1 e^{\lambda_1 t} + \mathbf{v}_2 e^{\lambda_2 t}, \qquad (8)$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of M and λ_1 and λ_2 are the associated eigenvalues.

The A, B terms that scale the complementary functions have been absorbed into the eigenvectors.

We shall only consider the possible forms of the phase portraits in the cases $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ (and $\lambda_1 \neq \lambda_2$).[2]

[2] For the degenerate case $\lambda_1 = \lambda_2 = \lambda$ (with λ real), in general $\mathsf{M} \neq \lambda\mathsf{I}$ and there is only a single eigenvector $\mathbf{v}$. The second solution is then of the form

$$\mathbf{Y}(t) = e^{\lambda t}(t\mathbf{v} + \mathbf{w}),$$

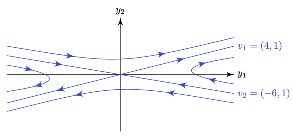
where the vector $\mathbf{w}$ satisfies $(\mathsf{M} - \lambda\mathsf{I})\mathbf{w} = \mathbf{v}$. Note that $\mathbf{w}$ is uniquely determined by this equation up to addition of multiples of $\mathbf{v}$, but such additional terms simply replicate the first solution. For the case when one of the eigenvalues vanishes, $\lambda_1 = 0$ say, the general solution is of the form

$$\mathbf{Y}(t) = \mathbf{v}_1 + \mathbf{v}_2 e^{\lambda_2 t}.$$

In the non-degenerate case, there are three distinct behaviours depending on the eigenvalues λ_1 and λ_2.

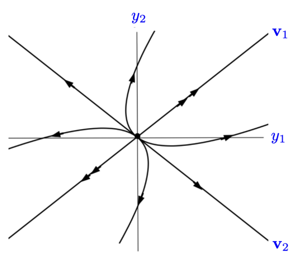
Case 1: λ_1 and λ_2 real and of opposite signs. Without loss of generality, we can take $\lambda_1 > 0$ and $\lambda_2 < 0$. The eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are necessarily real in this case. If $\mathbf{Y}$ starts out displaced from the origin along $\mathbf{v}_1$, it remains so and moves outwards as t increases (since $\lambda_1 > 0$). On the other hand, if $\mathbf{Y}$ starts out displaced along $\mathbf{v}_2$, it will move inwards along this direction approaching $\mathbf{Y} = 0$ as $t \to \infty$.

This case corresponds to a *saddle node*. An example phase portrait is shown to the right, corresponding to Eq. (7) with no forcing term. The arrows show the direction of evolution with increasing t. The curved trajectories in this figure can be added based on the flow direction along the eigenvectors. As $t \to \infty$, these curved lines become parallel to the eigenvector $\mathbf{v}_1$ with positive eigenvalue, while as $t \to -\infty$ they become parallel to $\mathbf{v}_2$.



Case 2: λ_1 and λ_2 real and of the same sign. Without loss of generality, we can take $|\lambda_1| > |\lambda_2|$. Again, the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are necessarily real and if $\mathbf{Y}$ starts out displaced along these it will continue so, moving outwards as t increases for $\lambda_1 > 0$ and inwards for $\lambda_1 < 0$.

This case corresponds to a *stable node* if λ_1 and $\lambda_2 < 0$, and an *unstable node* if λ_1 and $\lambda_2 > 0$. An unstable node is illustrated in the figure to the right. Here, a generic trajectory is parallel to $\mathbf{v}_1$ as $t \to \infty$ (as $\lambda_1 > \lambda_2$) and approaches the origin along $\mathbf{v}_2$ as $t \to -\infty$. In the unstable case, the directions of the arrows are reversed.



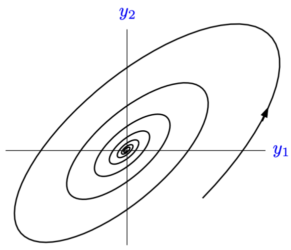
Case 3: λ_1 and λ_2 complex conjugate pairs. In this case, the eigenvectors are necessarily complex (as the matrix M is real) and are complex conjugates of each other: $\mathbf{v}_2 = \mathbf{v}_1^*$. Straight-line trajectrories are not possible.

The general solution, Eq. (8), for real $\mathbf{Y}$ can be written as

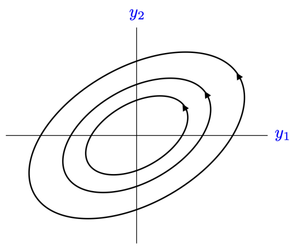
$$\begin{aligned}\mathbf{Y}(t) &= c\mathbf{v}_1 e^{\mathrm{Re}(\lambda_1)t} e^{i\mathrm{Im}(\lambda_1)t} + c^*\mathbf{v}_1^* e^{\mathrm{Re}(\lambda_1)t} e^{-i\mathrm{Im}(\lambda_1)t} \\ &= 2e^{\mathrm{Re}(\lambda_1)t}\{[c_1\mathrm{Re}(\mathbf{v}_1) - c_2\mathrm{Im}(\mathbf{v}_1)]\cos[\mathrm{Im}(\lambda_1)t] \\ &\qquad - [c_1\mathrm{Im}(\mathbf{v}_1) + c_2\mathrm{Re}(\mathbf{v}_1)]\sin[\mathrm{Im}(\lambda_1)t]\},\end{aligned}$$

where the complex constant $c = c_1 + ic_2$, with c_1 and c_2 real.

Trajectories generally spiral around the origin. If $\mathrm{Re}(\lambda_1) < 0$, we have a *stable spiral*, whereby the trajectories spiral into the origin as $t \to \infty$ (see the figure to the right for an example). For $\mathrm{Re}(\lambda_1) > 0$, we have an *unstable spiral* and the trajectories spiral outwards with increasing t.



However, if $\mathrm{Re}(\lambda_1) = 0$ we have a *centre* and the solutions are periodic giving closed trajectories in phase space. These are generally elliptical and have common centres at the origin (see figure to the right).



To find the sense of rotation, it is sufficient to determine $\dot{\mathbf{Y}}$ at one point. For example, if we evaluate $\dot{\mathbf{Y}}$ at $\mathbf{Y} = (1, 0)^T$, then $\dot{y}_2 > 0$ there implies counter-clockwise rotation.

5 Nonlinear dynamical systems

In this section, we briefly introduce systems of nonlinear differential equations. In particular, we shall see how the techniques of the previous section for systems of linear equations can be used to investigate the stability of equilibrium points of the nonlinear system.

Consider an *autonomous system* of two nonlinear, first-order differential equations:

$$\begin{aligned} \dot{x} &= f(x, y)\,, \\ \dot{y} &= g(x, y)\,. \end{aligned} \tag{9}$$

The functions $f(x, y)$ and $g(x, y)$ are general, nonlinear functions of the dependent variables x and y but are independent of time t (hence the system is autonomous).

Solving such systems of equations can be very difficult. However, we can learn a lot about the phase-space trajectories of the solutions of these equations by studying the equilibrium points and their stability.

5.1 Equilibrium points

Definition (Equilibrium point). An *equilibrium point* (or fixed point) of the system of equations (9) is a point at which $\dot{x} = \dot{y} = 0$.

If (x_0, y_0) is a fixed point of Eq. (9), this requires

$$f(x_0, y_0) = 0 \quad \text{and} \quad g(x_0, y_0) = 0\,.$$

We must solve these equations simultaneously to determine (x_0, y_0).

To determine the stability of an equilibrium point, we conduct a perturbation analysis. Let

$$x(t) = x_0 + \xi(t) \quad \text{and} \quad y(t) = y_0 + \eta(t)\,,$$

where ξ and η are small perturbations around the fixed point. Substituting into Eq. (9), we have, for example,

$$\begin{aligned} \dot{\xi} &= f(x_0 + \xi, y_0 + \eta) \\ &\approx f(x_0, y_0) + \xi \frac{\partial f}{\partial x}(x_0, y_0) + \eta \frac{\partial f}{\partial y}(x_0, y_0) \\ &\approx \xi \frac{\partial f}{\partial x}(x_0, y_0) + \eta \frac{\partial f}{\partial y}(x_0, y_0)\,. \end{aligned}$$

Here, we have performed a multivariate Taylor expansion and dropped higher-order terms. Similarly,

$$\dot{\eta} \approx \xi \frac{\partial g}{\partial x}(x_0, y_0) + \eta \frac{\partial g}{\partial y}(x_0, y_0)\,.$$

We can combine these linear equations into the vector equation

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \tag{10}$$

where the matrix of first derivatives is evaluated at (x_0, y_0). This is a linear system of first-order differential equations and we can apply the techniques of Sec. 4 to determine the nature of the equilibrium point. In particular, the stability is determined by the eigenvalues of the matrix of first derivatives.

Example (Population dynamics: predator–prey system). Consider an ecosytem with predators and prey. Let the number of prey at time t be $x(t)$ and the number of predators be $y(t)$. We model the dynamics of the prey as

$$\dot{x} = \alpha x - \beta x^2 - \gamma x y\,, \tag{11}$$

where α, β and γ are positive constants. In the absence of predators ($y = 0$), this is the logistic differential equation of Topic III, where, recall, α describes the excess rate of births over natural deaths and the term $-\beta x^2$ increases the death rate at high x to account for competition over some scarce resource. The term $-\gamma x y$ in Eq. (11) accounts for the prey being killed by the predators; we assume that the predators have infinite appetite, so consume all prey that they encounter.

We model the dynamics of the predators as

$$\dot{y} = \epsilon x y - \delta y\,,$$

where ϵ and δ are further positive constants. The first term on the right is the birth rate of predators, which increases if more prey is available to sustain the population. The final term is the natural death rate of the predators. If there are no prey ($x = 0$), the number of predators decays exponentially.

We shall consider the following specific example:

$$\begin{aligned} \dot{x} &= 8x - 2x^2 - 2xy\,, \\ \dot{y} &= xy - y\,. \end{aligned} \tag{12}$$

The equilibrium points of this nonlinear, first-order autonomous system are where

$$2x(4 - x - y) = 0 \quad \text{and} \quad y(x-1) = 0\,.$$

The first equation requires either $x = 0$ or $x = 4 - y$. In the former case, the second equation then requires $y = 0$ so we have an equilibrium point at $(0, 0)$. On the other hand, if $x = 4 - y$, the second equation reduces to

$$y(3 - y) = 0\,,$$

so either $y = 0$ or $y = 3$. We thus have two further equilibrium points: $(4, 0)$ and $(1, 3)$.

We consider the stability of these in turn using Eq. (10). Noting that

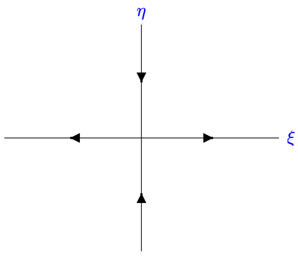
$$f(x, y) = 8x - 2x^2 - 2xy \quad \text{and} \quad g(x, y) = xy - y\,,$$

the required derivatives evaluate to

$$\begin{aligned} f_x &= 8 - 4x - 2y & f_y &= -2x\,, \\ g_x &= y & g_y &= x - 1\,. \end{aligned}$$

(0, 0). Perturbations around this point evolve as

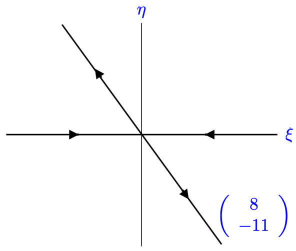
$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$



The eigenvalues of the matrix are clearly 8 and -1 and associated eigenvectors are $(1, 0)^T$ and $(0, 1)^T$. As the eigenvalues are real and of opposite sign, the equilibrium point is a saddle node. Perturbations along the x-direction move away from $(0, 0)$, while those along the y-direction move towards it (see figure to the right). Note that if motion is restricted to $y = 0$, we recover the unstable nature of $x = 0$ for the logistic differential equation .

$(\mathbf{4}, \mathbf{0})$. Perturbations around this point evolve as

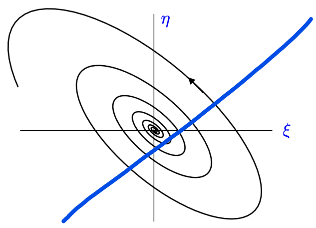
$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} -8 & -8 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$



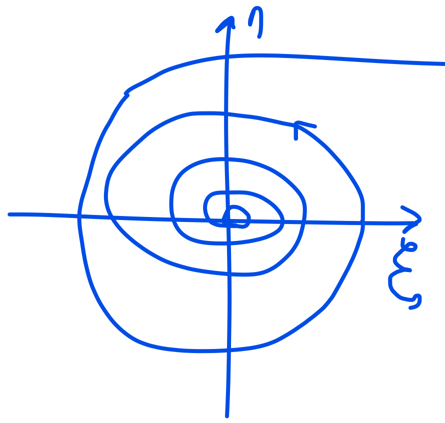
The eigenvalues of the matrix are -8 and 3 and the eigenvectors are $(1,0)^T$ and $(8,-11)^T$, respectively. We see that this is also a saddle node, with displacements along the x-direction moving back towards the equilibrium point, but those along $(8,-11)^T$ moving away. For motion restricted to $y = 0$, we recover the stable nature of the equilibrium point $x = 4$ for the logistic differential equation.

$(\mathbf{1}, \mathbf{3})$. Finally, perturbations around this point evolve as

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$



The eigenvalues of the matrix are $-1 \pm i\sqrt{5}$. Since these are a complex-conjugate pair with negative real part, the equilibrium point is a stable spiral. We can determine the sense of rotation by considering $(\xi, \eta) = (1, 0)$. For such a displacement, $\dot{\eta} = 3$. Since this is positive, the spiral is traversed anti-clockwise (see figure to the right).



The full phase portrait is shown in Fig. 3. The equilibrium (saddle) points at $(0,0)$ and $(4,0)$ are unstable and the introduction of any predators around these points will drive the system to spiral towards the stable equilibrium point at $(1,3)$.

6 Partial differential equations

In this final section, we very briefly introduce *partial differential equations*, where we have multiple independent variables.

We shall illustrate some of the key ideas using the simple examples of the wave equation and the diffusion equation.

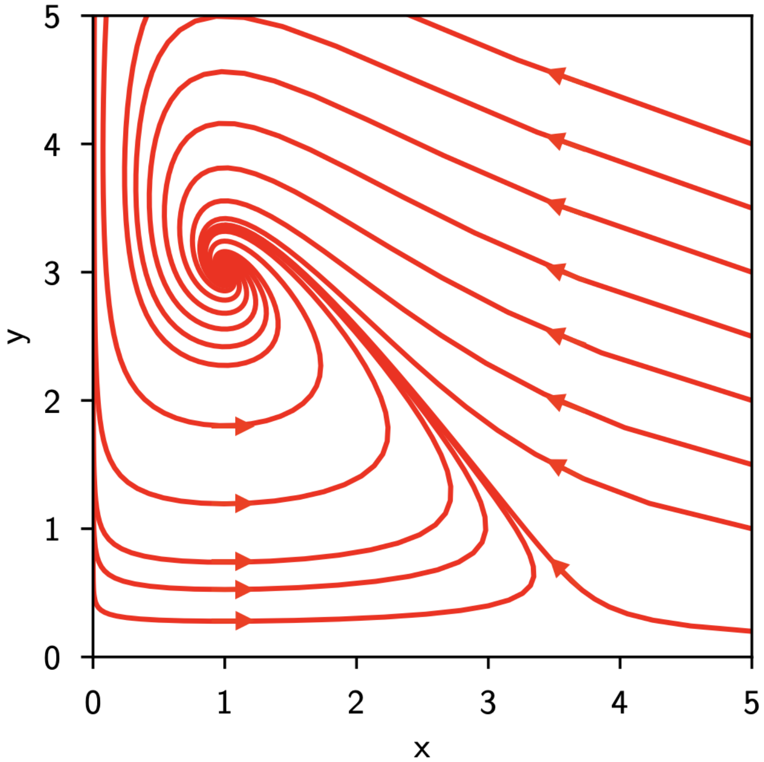


Figure 3: Phase portrait for the predator–prey system described by Eq. (12). Note the stable spiral equilibrium point at $(1, 3)$, where the number of prey $x = 1$ and predators $y = 3$.

6.1 First-order wave equation

Consider a function $\psi(x,t)$, depending on position x and time t, which satisfies the *first-order wave equation*

$$\frac{\partial\psi}{\partial t} - c\frac{\partial\psi}{\partial x} = 0\,. \tag{13}$$

Here, c is a constant with dimensions of speed. We shall see that this equation describes waves that propagate to the left with speed c.

We attempt to solve Eq. (13) with the *method of characteristics*. To see how this works, consider how ψ varies along the path $x(t)$ so that $\psi = \psi(x(t),t)$ can be considered a function of t. Using the multivariate chain rule, we have

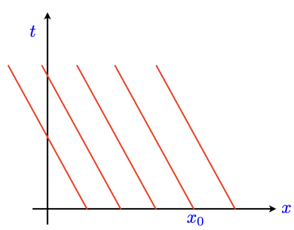
$$\begin{aligned}\frac{d\psi}{dt} &= \frac{\partial\psi}{\partial t} + \frac{\partial\psi}{\partial x}\frac{dx}{dt}\\ &= \frac{\partial\psi}{\partial x}\left(c + \frac{dx}{dt}\right)\,,\end{aligned}$$

where we have used Eq. (13) to substitute for $\partial\psi/\partial t$.

If we choose a path with

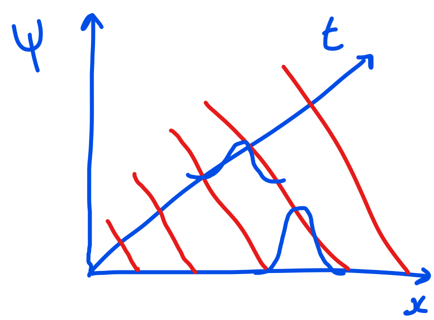
$$\frac{dx}{dt} = -c\,,$$

we see that $d\psi/dt = 0$ along the path and so ψ is constant. These paths are given by



$$x(t) = x_0 - ct\,,$$

where $\{x_0\}$ labels the paths. These paths are called the *characteristics* of the partial differential equation (13) and are illustrated in the figure to the right.



Left-moving wavelike solution

As ψ is constant along the characteristics, the general solution is $\psi(x,t) = f(x_0)$, where $x_0 = x + ct$, for some arbitrary function f. This gives

$$\psi(x,t) = f(x+ct)\,. \tag{14}$$

As t increases, we are simply taking the x-dependence of ψ at $t=0$ and translating it to the left by ct. We see that Eq. (14) is the solution of our partial differential equation (13) with initial condition $\psi(x, t = 0) = f(x)$.

Example (unforced wave equation). Consider

$$\frac{\partial \psi}{\partial t} - c\frac{\partial \psi}{\partial x} = 0 \qquad \text{with} \qquad \psi(x, 0) = x^2 - 3\,.$$

The general solution of the partial differential equation is $\psi(x, t) = f(x + ct)$ and imposing the initial condition gives

$$\rightarrow \psi(x,0) = f(x) = x^2 - 3.$$

$$\psi(x, t) = (x + ct)^2 - 3\,.$$

Example (forced wave equation). We can also use the method of characteristics to solve forced wave equations such as

$$\frac{\partial \psi}{\partial t} + 5\frac{\partial \psi}{\partial x} = e^{-t} \qquad \text{with} \qquad \psi(x, 0) = e^{-x^2}\,.$$

The characteristics have $dx/dt = 5$, so $x = x_0 + 5t$. Along these, the partial differential equation reduces to

$$\frac{d\psi}{dt} = e^{-t}\,,$$

with solution

$$\psi = f(x_0) - e^{-t}\,,$$

for some arbitrary function $f(x_0)$. Finally, imposing the boundary condition and noting that $x = x_0$ at $t = 0$, we have

$$\psi(x, 0) = f(x_0) - 1 = e^{-x_0^2} \quad \Rightarrow \quad f(x_0) = 1 + e^{-x_0^2}\,,$$

so that

$$\psi(x, t) = 1 + e^{-(x-5t)^2} - e^{-t}\,.$$

6.2 Second-order wave equation

The first-order wave equation admits solutions that propagate in only one direction. In practice, most systems supporting wavelike solutions allow these to propagate

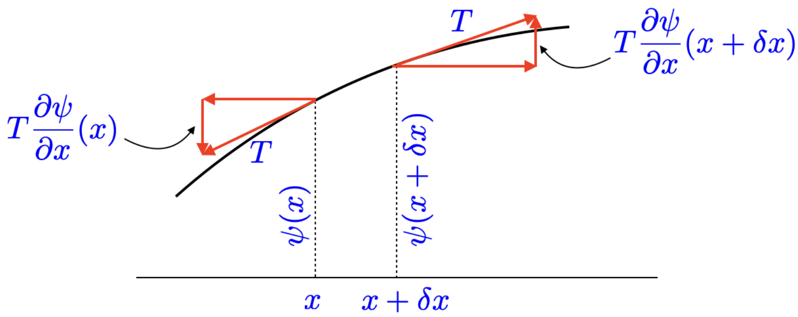


Figure 4: Forces acting on a short segment of length δx of a string held under tension T, when the sring is displaced vertically by $\psi(x,t)$.

in either direction. The relevant partial differential equation is then the *second-order wave equation*:

$$\frac{\partial^2\psi}{\partial t^2} - c^2\frac{\partial^2\psi}{\partial x^2} = 0\,. \tag{15}$$

Aside: waves on a string (non-examinable)

As an example of a system described by the second-order wave equation, consider the vertical displacement $\psi(x,t)$ of a string held under tension T. A short length δx of the string at x is pulled by the tension at each of its ends, with the forces acting along the tangents to the string there. Assuming the slope of the string is always small and the tension remains at T, the vertical component of the force at x from the string to the right is approximately $T\partial\psi/\partial x$; see Fig. 4.

The net vertical force on δx is approximately

$$T\frac{\partial\psi}{\partial x}(x+\delta x,t) - T\frac{\partial\psi}{\partial x}(x,t) = T\delta x\frac{\partial^2\psi}{\partial x^2}(x,t) + O(\delta x^2)\,.$$

This must equal the product of the mass of the string segment and its acceleration $\partial^2\psi/\partial t^2$. If the mass per unit length is μ, we therefore have

$$T\delta x\frac{\partial^2\psi}{\partial x^2} + O(\delta x^2) = \mu\delta x\frac{\partial^2\psi}{\partial t^2}\,.$$

Dividing by δx and taking the limit $\delta x\to 0$, we have

$$\frac{\partial^2\psi}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2\psi}{\partial x^2}\,,$$

which is a wave equation with wave speed $c=\sqrt{T/\mu}$.

To solve the second-order wave equation (15), we note that it can be factored as

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\psi = 0$$

since partial derivatives commute. The two operators here commute so both $f(x+ct)$, which is nulled by the first factor, and $g(x-ct)$, nulled by the second, are solutions, where f and g are arbitrary functions. As the equation is linear, the superposition of these is also a solution. We thus have the general solution of the second-order wave equation:

$$\psi(x,t) = f(x+ct) + g(x-ct)\,. \tag{16}$$

This is a superposition of left- and right-moving waves.

We can show that this is the most general solution by changing independent variables from x and t to

$$\xi = x + ct \qquad \text{and} \qquad \eta = x - ct\,.$$

The coordinates ξ and η are constants along the characteristics of $\partial/\partial t - c\partial/\partial x$ and $\partial/\partial t + c\partial/\partial x$, respectively. Using the multivariate chain rule, we have

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\,,$$
$$\frac{\partial}{\partial t} = c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta}\,.$$

$$\frac{\partial}{\partial x} = \underbrace{\frac{\partial \xi}{\partial x}\Big|_t}_{1} \frac{\partial}{\partial \xi} + \underbrace{\frac{\partial \eta}{\partial x}\Big|_t}_{1} \frac{\partial}{\partial \eta}$$

It follows that

$$\frac{\partial}{\partial t} - c\frac{\partial}{\partial x} = -2c\frac{\partial}{\partial \eta}\,,$$
$$\frac{\partial}{\partial t} + c\frac{\partial}{\partial x} = 2c\frac{\partial}{\partial \xi}\,,$$

so that

$$0 = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\psi = -4c^2\frac{\partial^2\psi}{\partial\eta\partial\xi}\,.$$

Integrating twice, the most general solution of this is

$$\psi = f(\xi) + g(\eta)\,,$$

$$\int \frac{\partial^2\psi}{\partial\eta\partial\xi}\,d\xi = \int 0\,d\xi$$
$$\frac{\partial\psi}{\partial\eta} = h(\eta)$$
$$\int \frac{\partial\psi}{\partial\eta}\,d\eta = \int h(\eta)\,d\eta$$
$$\psi = f(\xi) + \int h(\eta)\,d\eta$$
$$= f(\xi) + g(\eta)$$

in agreement with Eq. (16).

Example. Consider

$$\frac{\partial^2\psi}{\partial t^2} - c^2\frac{\partial^2\psi}{\partial x^2} = 0\,,$$

with initial conditions[3] at $t = 0$

$$\psi(x,0) = \frac{1}{1+x^2} \quad \text{and} \quad \frac{\partial\psi}{\partial t}(x,0) = 0\,.$$

This might describe a long string displaced vertically by $1/(1+x^2)$ at $t = 0$ and released from rest.

The general solution of the wave equation is

$$\psi(x,t) = f(x+ct) + g(x-ct)\,.$$

Evaluating this at $t = 0$, we must have

$$f(x) + g(x) = \frac{1}{1+x^2}\,. \tag{17}$$

The second initial condition gives

$$\frac{\partial\psi}{\partial t}(x,0) = cf'(x) - cg'(x) = 0\,,$$

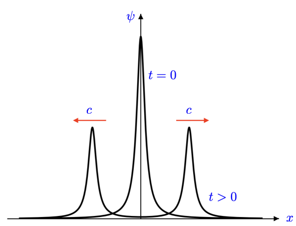
so that $f(x) = g(x) + A$, where A is a constant. Combining with Eq. (17), we have

$$f(x) = \frac{1}{2\,(1+x^2)} + \frac{A}{2}\,,$$
$$g(x) = \frac{1}{2\,(1+x^2)} - \frac{A}{2}\,.$$

The constant A cancels on forming $f(x+ct)+g(x-ct)$ to leave the solution

$$\psi(x,t) = \frac{1}{2}\left[\frac{1}{1+(x+ct)^2} + \frac{1}{1+(x-ct)^2}\right]\,.$$

Note how the initial disturbance separates into a left- and right-moving wave, each with half the amplitude of the initial disturbance, propagating at speed c; see the figure to the right.



[3]A second-order ordinary differential equation needs two initial conditions, say the initial value of the dependent variable and its derivative. The second-order wave equation is second-order in time derivatives and so we need to give ψ and its time derivative $\partial\psi/\partial t$ *at each value of* x to specify the solution.

6.3 The diffusion equation

A second important example of a partial differential equation is the *diffusion equation.* For example, the temperature T of a one-dimensional system evolves due to thermal conduction (diffusion of thermal energy) as

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}\,. \tag{18}$$

The positive quantity κ is a constant and is known as the (thermal) diffusion coefficient.

Example. The temperature $T(x,t)$ in an infinitely long rod with one end at $x = 0$ satisfies the diffusion equation (18). For $t > 0$, the temperature at $x = 0$ is maintained at 1. As $t \to 0_+$, $T = 0$ for $x > 0$ so the rod is initially cold except at $x = 0$.

It is possible to find *similarity solutions* of the form $T(x,t) = \Theta(\eta)$, where

$$\eta \equiv \frac{x}{2\sqrt{\kappa t}} \qquad (t > 0)\,.$$

The form of η is motivated by dimensional analysis of the diffusion equation. Using the chain rule, we have

$$\frac{\partial T}{\partial t} = \frac{\partial \eta}{\partial t}\frac{d\Theta}{d\eta} = -\frac{\eta}{2t}\frac{d\Theta}{d\eta}$$

and

$$\frac{\partial T}{\partial x} = \frac{\partial \eta}{\partial x}\frac{d\Theta}{d\eta} = \frac{1}{2\sqrt{\kappa t}}\frac{d\Theta}{d\eta}$$

so

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{2\sqrt{\kappa t}}\frac{\partial \eta}{\partial x}\frac{d^2\Theta}{d\eta^2} = \frac{1}{4\kappa t}\frac{d^2\Theta}{d\eta^2}\,.$$

The diffusion then becomes an ordinary differential equation for $\Theta(\eta)$:

$$\frac{d^2\Theta}{d\eta^2} + 2\eta\frac{d\Theta}{2\eta} = 0\,. \tag{19}$$

Equation (19) is a separable first-order equation for $d\Theta/d\eta$ and is solved by

$$\frac{d\Theta}{d\eta} = Ae^{-\eta^2}\,,$$

where A is a constant. Integrating again gives

$$\Theta(\eta) = A\int_0^\eta e^{-u^2}\,du + B\,,$$

where B is a further constant. The integral here can be written in terms of the *error function*, defined by

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du\,;$$

it satisfies $\mathrm{erf}(x) \to 1$ as $x \to \infty$. We have

$$\Theta(\eta) = C\,\mathrm{erf}(\eta) + B\,,$$

with $C = \sqrt{\pi}A/2$.

We now impose the boundary conditions. As $t \to 0_+$ for $x > 0$, we have $\eta \to \infty$. We require $T \to 0$ in this limit, so $B + C = 0$. At $x = 0$ for $t > 0$, we have $\eta = 0$. Here, $T = 1$ so we require $B = 1$. It follows that the complete solution for $t > 0$ is

$$T(x,t) = 1 - \mathrm{erf}\left(\frac{x}{2\sqrt{\kappa t}}\right)\,.$$

This solution is shown for several values of t in the figure to the right. Note how the temperature tends to uniform, $T = 1$, as $t \to \infty$.